13 Pontryagin's Maximum Principle

We explain Pontryagin's maximum principle and give some examples of its use.

13.1 Heuristic derivation

Pontryagin's maximum principle (PMP) states *a necessary condition that must hold on an optimal trajectory*. It is a calculation for a *fixed* initial value of the state, $x(0)$. In comparison, the DP approach is a calculation for a general initial value of the state. PMP can be used as both a computational and analytic technique (and in the second case can solve the problem for general initial value.)

Consider first a time-invariant formulation, with plant equation $\dot{x} = a(x, u)$, instantaneous cost $c(x, u)$, stopping set $\mathcal{S}$ and terminal cost $\mathbf{K}(x)$. The value function $F(x)$ obeys the DP equation (without discounting)

$$\inf_{u \in \mathcal{U}} \left[c(x, u) + \frac{\partial F}{\partial x} a(x, u) \right] = 0, \tag{13.1}$$

outside $\mathcal{S}$, with terminal condition

$$F(x) = \mathbf{K}(x), \quad x \in \mathcal{S}. \tag{13.2}$$

Define the **adjoint variable**

$$\lambda = -F_x \tag{13.3}$$

This is a column n-vector, and is to be regarded as a function of time on the path. The proof that F_x exists in the required sense is actually a tricky technical matter. Also define the **Hamiltonian**

$$H(x, u, \lambda) = \lambda^\top a(x, u) - c(x, u), \tag{13.4}$$

a scalar, defined at each point of the path as a function of the current x, u and λ.)

Theorem 13.1 (PMP) *Suppose $u(t)$ and $x(t)$ represent the optimal control and state trajectory. Then there exists an adjoint trajectory $\lambda(t)$ such that together $u(t)$, $x(t)$ and $\lambda(t)$ satisfy*

$$\dot{x} = H_\lambda, \qquad [= a(x, u)] \tag{13.5}$$

$$\dot{\lambda} = -H_x, \qquad [= -\lambda^\top a_x + c_x] \tag{13.6}$$

and for all t, $0 \le t \le T$, and all feasible controls v,

$$H(x(t), v, \lambda(t)) \le H(x(t), u(t), \lambda(t)), \tag{13.7}$$

i.e., the optimal control $u(t)$ is the value of v maximizing $H((x(t), v, \lambda(t))$.

'Proof.' Our heuristic proof is based upon the DP equation; this is the most direct and enlightening way to derive conclusions that may be expected to hold in general.

Assertion (13.5) is immediate, and (13.7) follows from the fact that the minimizing value of u in (13.1) is optimal. We can write (13.1) in incremental form as

$$F(x) = \inf_u [c(x, u)\delta + F(x + a(x, u)\delta)] + o(\delta).$$

Using the chain rule to differentiate with respect to x_i yields

$$-\lambda_i(t) = \frac{\partial c}{\partial x_i}\delta - \lambda_i(t + \delta) - \sum_j \frac{\partial a_j}{\partial x_i} \lambda_j(t + \delta) + o(\delta)$$

whence (13.6) follows. ■

Notice that (13.5) and (13.6) each give n equations. Condition (13.7) gives a further m equations (since it requires stationarity with respect to variation of the m components of u.) So in principle these equations, if nonsingular, are sufficient to determine the $2n + m$ functions $u(t)$, $x(t)$ and $\lambda(t)$.

One can make other assertions, including specification of end-conditions (the so-called **transversality conditions**.)

Theorem 13.2 *(i) $H = 0$ on the optimal path. (ii) The sole initial condition is specification of the initial x. The terminal condition*

$$(\lambda + \mathbf{K}_x)^\top \sigma = 0 \tag{13.8}$$

holds at the terminal x for all σ such that $x + \epsilon\sigma$ is within $o(\epsilon)$ of the termination point of a possible optimal trajectory for all sufficiently small positive ϵ.

'Proof.' Assertion (i) follows from (13.1), and the first assertion of (ii) is evident. We have the terminal condition (13.2), from whence it follows that $(F_x - \mathbf{K}_x)^\top \sigma = 0$ for all x, σ such that x and $x + \epsilon\sigma$ lie in $\mathcal{S}$ for all small enough positive ϵ. However, we are only interested in points where an optimal trajectory makes it first entry to $\mathcal{S}$ and at these points (13.3) holds. Thus we must have (13.8). ■

13.2 Example: bringing a particle to rest in minimal time

A particle with given initial position and velocity $x_1(0)$, $x_2(0)$ is to be brought to rest at position 0 in minimal time. This is to be done using the control force u, such that $|u| \le 1$, with dynamics of $\dot{x}_1 = x_2$ and $\dot{x}_2 = u$. That is,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{13.9}$$

and we wish to minimize

$$\mathbf{C} = \int_0^T 1\, dt$$

where T is the first time at which $x = (0, 0)$. The Hamiltonian is

$$H = \lambda_1 x_2 + \lambda_2 u - 1\,,$$

which is maximized by $u = \text{sign}(\lambda_2)$. The adjoint variables satisfy $\dot\lambda_i = -\partial H/\partial x_i$, so

$$\dot\lambda_1 = 0, \qquad \dot\lambda_2 = -\lambda_1. \tag{13.10}$$

The terminal x must be 0, so in (13.8) we can only take $\sigma = 0$ and so (13.8) provides no additional information for this problem. However, if at termination $\lambda_1 = \alpha$, $\lambda_2 = \beta$, then in terms of time to go we can compute

$$\lambda_1 = \alpha, \qquad \lambda_2 = \beta + \alpha s.$$

These reveal the form of the solution: there is at most one change of sign of λ_2 on the optimal path; u is maximal in one direction and then possibly maximal in the other.

Appealing to the fact that $H = 0$ at termination (when $x_2 = 0$), we conclude that $|\beta| = 1$. We now consider the case $\beta = 1$. The case $\beta = -1$ is similar.

If $\beta = 1$, $\alpha \geq 0$ then $\lambda_2 = 1 + \alpha s \geq 0$ for all $s \geq 0$ and

$$u = 1, \qquad x_2 = -s, \qquad x_1 = s^2/2.$$

In this case the optimal trajectory lies on the parabola $x_1 = x_2^2/2$, $x_1 \geq 0, x_2 \leq 0$. This is half of the **switching locus** $x_1 = \pm x_2^2/2$.

If $\beta = 1$, $\alpha < 0$ then $u = -1$ or $u = 1$ as the time to go is greater or less than $s_0 = 1/|\alpha|$. In this case,

$$\begin{array}{llll} u = -1, & x_2 = (s - 2s_0), & x_1 = 2s_0 s - \frac{1}{2}s^2 - s_0^2\,, & s \geq s_0\,, \\ u = 1, & x_2 = -s, & x_1 = \frac{1}{2}s^2, & s \leq s_0\,. \end{array}$$

The control rule expressed as a function of s is open-loop, but in terms of (x_1, x_2) and the switching locus, it is closed-loop.

Notice that the path is sensitive to the initial conditions, in that the optimal path is very different for two points just either side of the switching locus.

13.3 Connection with Lagrangian multipliers

An alternative way to understand the maximum principle is to think of λ as a Lagrangian multiplier for the constraint $\dot x = a(x, u)$. Consider the Lagrangian form

$$L = \int_0^T \left[-c - \lambda^\top(\dot x - a)\right] dt - \mathbf{K}(x(T))\,,$$

to be maximized with respect to the (x, u, λ) path. Here $x(t)$ first enters a stopping set $\mathcal{S}$ at $t = T$. We integrate $\lambda^\top \dot x$ by parts to obtain

$$L = -\lambda(T)^\top x(T) + \lambda(0)^\top x(0) + \int_0^T \left[\dot\lambda^\top x + \lambda^\top a - c\right] dt - \mathbf{K}(x(T))\,.$$

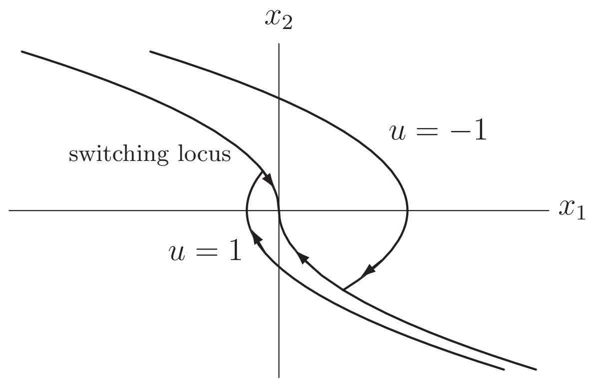

Figure 2: Optimal trajectories for the problem

The integrand must be stationary with respect to $x(t)$ and hence $\dot\lambda = -H_x$, i.e., (13.6). The expression must also be stationary with respect to $\epsilon > 0$, $x(T) + \epsilon\sigma \in \mathcal{S}$ and hence $(\lambda(T) + \mathbf{K}_x(x(T)))^\top \sigma = 0$, i.e., (13.8). It is good to have this alternative view, but the treatment is less immediate and less easy to rigorise.

13.4 Example: use of the transversality conditions

If the terminal time is constrained then (as we see in the next lecture) we no longer have Theorem 13.2 (i), i.e., that H is maximized to 0, but the other claims of Theorems 13.1 and 13.2 continue to hold.

Consider the a problem with the dynamics (13.9), but with u unconstrained, $x(0) = (0, 0)$ and cost function

$$\mathbf{C} = \tfrac{1}{2}\int_0^T u(t)^2\, dt - x_1(T)$$

where T is fixed and given. Here $K(x) = -x_1(T)$ and the Hamiltonian is

$$H(x, u, \lambda) = \lambda_1 x_2 + \lambda_2 u - \tfrac{1}{2}u^2,$$

which is maximized at $u(t) = \lambda_2(t)$. Now $\dot\lambda_i = -\partial H/\partial x_i$ gives

$$\dot\lambda_1 = 0, \qquad \dot\lambda_2 = -\lambda_1.$$

In the terminal condition, $(\lambda + \mathbf{K}_x)^\top \sigma = 0$, σ is arbitrary and so we also have

$$\lambda_1(T) - 1 = 0, \qquad \lambda_2(T) = 0.$$

Thus the solution must be $\lambda_1(t) = 1$ and $\lambda_2(t) = T - t$. Hence the optimal applied force is $u(t) = T - t$, which decreases linearly with time and reaches zero at T.