Random walks and branching random walks: old and new perspectives

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1 Preliminaries

Let $(X_n)_{n\geq 0}$ be a simple random walk in $\mathbb{Z}^d$ with $d \geq 1$. Note that the time index will always be $\mathbb{N}$. For every $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}^d$ we write $P^n(x,y) = p_n(x,y) = \mathbb{P}_x(X_n = y)$ for the n-step transition probability from x to y. Note that if $x = 0$ we will sometimes omit it from the notation and we will simply write $p_n(y)$ for $p_n(0,y)$. Note that by translation invariance of the walk we have $p_n(x,y) = p_n(0, x-y)$. We say that n and x are of the same parity if $p_n(x) > 0$. For every n and x we write

$$\overline{p}_n(x) = 2 \cdot \left(\frac{d}{2\pi n}\right)^{d/2} \cdot \exp\left(-\frac{d\|x\|^2}{2n}\right).$$

In dimension one, a direct calculation using Stirling's formula immediately yields the following:

Exercise 1.1. Let X be a simple symmetric random walk on $\mathbb{Z}$ starting from 0. Show that for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $m \leq \sqrt{n}$ we have

$$\mathbb{P}_0(X_{2n} = 2m) = \overline{p}_{2n}(2m)(1 + o(1)) \text{ as } n \to \infty.$$

Hint: Recall Stirling's formula $n! \sim n^n\sqrt{2\pi n} \cdot e^{-n}$ as $n \to \infty$.

In higher dimensions, one can also obtain an analogous result, but it is more tedious. We state without proof the local CLT that we will use very often in this course.

Theorem 1.2. (Local CLT [10, Proposition 1.2.5]) *Let $d \geq 1$ and let X be a simple random walk in $\mathbb{Z}^d$ started from 0. Suppose that n and x are of the same parity. Let $\alpha < 2/3$. If $\|x\| \leq n^\alpha$, then*

$$p_n(x) = \overline{p}_n(x) \cdot \left(1 + O(n^{3\alpha-2})\right).$$

Exercise 1.3. Let X be a simple random walk on $\mathbb{Z}^d$ started from 0. Show using Azuma's inequality or otherwise that there exist positive constants c_1 and c_2 so that for all $x \in \mathbb{Z}^d$

$$\mathbb{P}_0(X_n = x) \leq c_1 \exp(-c_2\|x\|^2/n).$$

Notation: For functions $f, g : \mathbb{N} \to \mathbb{R}_+$ we write $f \lesssim g$ if there exists a positive constant C so that for all $n \in \mathbb{N}$ we have $f(n) \leq Cg(n)$. We write $f \gtrsim g$ if $g \lesssim f$. Finally we write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

By the local CLT we see that

$$p_{2n}(0) \asymp \frac{1}{n^{d/2}}, \tag{1.1}$$

and hence we recover Polya's theorem: when $d \leq 2$, the SRW is recurrent, while when $d \geq 3$ it is transient.

Exercise 1.4. Prove (1.1) using Stirling's formula and the concentration of a Binomial random variable of parameters n and $1/d$ around its mean.

In the transient regime, we now define the Green's function as follows: for $x, y \in \mathbb{Z}^d$ with $d \geq 3$

$$g(x,y) = \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbf{1}(X_n = y)\right] = \sum_{n=0}^{\infty} p_n(x,y).$$

When $x = 0$ we simply write $g(y)$ for $g(0, y)$. By conditioning on the first step of the random walk it is immediate to deduce the following:

Lemma 1.5. *The Green's function $g : \mathbb{Z}^d \to \mathbb{R}_+$ is harmonic on $\mathbb{Z}^d \setminus \{0\}$.*

Exercise 1.6. Using reversibility and the Cauchy-Schwartz inequality prove that for all $x \in \mathbb{Z}^d$

$$p_{2n}(0,x) \leq \sqrt{p_{2n}(0,0)p_{2n}(x,x)} = p_{2n}(0,0). \tag{1.2}$$

Combining (1.1) together with (1.2) we see that the Green's function is well defined when $d \geq 3$, as we get a converging series.

The following asymptotic expression for the Green's function will be used throughout these notes.

Theorem 1.7 (Spitzer)**.** *For all $d \geq 3$ and $\alpha < d$ as $\|x\| \to \infty$ we have*

$$g(x) = \frac{c(d)}{\|x\|^{d-2}} + o(\|x\|^{-\alpha}), \quad \textit{where} \;\; c(d) = \frac{d}{2}\Gamma(d/2-1)\pi^{-d/2}.$$

Exercise 1.8. Prove Spitzer's result using the local CLT and by approximating the Riemann sum by an integral (see [11, Lemma 12.1.1]).

In the following exercise we obtain an upper bound on $g(x)$ of the correct order but without the sharp constant provided to us by Spitzer's result, which in turn follows by the local CLT.

Exercise 1.9. Let X be a SRW in $\mathbb{Z}^d$ with $d \geq 1$. Without appealing to the local CLT establish the following:

1. For all x of the same parity as 0 and satisfying $\|x\| \leq \sqrt{n}$ prove that
$$p_n(0,x) \asymp \frac{1}{n^{d/2}}.$$
2. Using reversibility prove that
$$\mathbb{P}_0(X_n = x) \leq 2 \cdot \mathbb{P}_0\big(X_n = x, \|X_{\lfloor n/2 \rfloor}\| \geq \|x\|/2\big)\,.$$
3. Using the above and Azuma's inequality, show that there exist positive constants c_1 and c_2 such that for all x
$$\mathbb{P}_0(X_n = x) \leq \frac{c_1}{n^{d/2}} \exp(-c_2\|x\|^2/n).$$
4. Combining all of the above show that
$$g(x) \asymp \|x\|^{2-d}.$$

For a set $A \subseteq \mathbb{Z}^d$ we write

$$H_A = \inf\{n \geq 0 : X_n \in A\} \quad \text{and} \quad \widetilde{H}_A = \inf\{n \geq 1 : X_n \in A\}$$

for the first hitting and first return time to A respectively.

For a finite set A with $x \in A$, we write

$$g_A(x, y) = \mathbb{E}_x\left[\sum_{j=0}^{H_{A^c}} \mathbf{1}(X_j = y)\right]$$

We write $B(0, n) = \{x \in \mathbb{Z}^d : \|x\| < n\}$ for the Euclidean lattice ball of radius n.

Lemma 1.10. *Let $x \in B(0, n/4)$ and $T = \inf\{j \geq 0 : X_j \in \partial B(0, n)\}$. Then for all $y \in \partial B(0, n)$ we have*

$$\mathbb{P}_x(X_T = y) \asymp \frac{1}{n^{d-1}},$$

where the constants appearing in $\asymp$ are universal over all n.

Sketch of proof. Let $\zeta = \inf\{j \geq 1 : X_j \in \{0\} \cup \partial B(0, n)\}$. Then check that

$$\mathbb{P}_x(X_T = y) = g_{B(0,n)}(0, 0)\mathbb{P}_y(X_\zeta = 0)\,.$$

Then it suffices to show that $\mathbb{P}_y(X_\zeta = 0) \asymp n^{1-d}$, as $g_{B(0,n)}(0, 0) \asymp 1$. To prove this, we define an intermediate scale, i.e. we first wait for the walk to either hit $B(0, n - 3)$ or exit $B(0, n)$. We then require the walk to be at $B(0, n - 3)$ at this time and estimate the probability that starting from there the walk hits 0 before hitting $\partial B(0, n)$. Finally to achieve this, we use the harmonicity of the Green's function g. □

Theorem 1.11 (Harnack inequality)**.** *Let $f : B(0, n) \to \mathbb{R}_+$ be a harmonic function in $B(0, n-1)$. Then for all $0 < r < 1$, there exists a positive constant $C = C_r$ so that*

$$\sup_{x \in B(0,rn)} f(x) \leq C \inf_{x \in B(0,rn)} f(x).$$

2 Intersections of random walks

In this section we will study the question of intersections of independent simple random walks on $\mathbb{Z}^d$ for all d. We start with the question of collisions to see the analogy.

So let X and Y be two independent simple random walks on $\mathbb{Z}^d$ for $d \geq 1$ starting from 0. Let

$$C_n = \sum_{i=0}^{n} \mathbf{1}(X_i = Y_i).$$

Then taking expectations of both sides we get

$$\mathbb{E}[C_n] = \sum_{i=0}^{n} \mathbb{P}_0(X_{2i} = 0) \asymp \sum_{i=1}^{2n} \frac{1}{i^{d/2}} \asymp \begin{cases} \sqrt{n} \text{ if } d = 1 \\ \log n \text{ if } d = 2 \\ 1 \text{ if } d \geq 3. \end{cases}$$

We thus see from here that dimension 2 is critical for the question of collisions which is of course the well-known theorem of Polya.

Next we move on to intersections. Let I_n be the total number of intersections of X and Y up to time n, i.e.

$$I_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbb{1}(X_i = Y_j).$$

Taking expectations above we get

$$\mathbb{E}[I_n] = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbb{P}_0(X_{i+j} = 0) \asymp \sum_{i=0}^{2n} i \cdot p_i(0,0).$$

Using the LCLT, we now get that

$$\mathbb{E}[I_n] \asymp \sum_{i=1}^{n} \frac{1}{i^{\frac{d}{2}-1}} \asymp \begin{cases} \sqrt{n} \text{ if } d = 3 \\ \log n \text{ if } d = 4 \\ 1 \text{ if } d \geq 5. \end{cases}$$

We see thus that dimension 4 is the critical dimension when considering intersections analogously to dimension 2 being the critical dimension when considering collisions.

In the next section we are going to calculate the probability that one random walk avoids a two sided random walk in four dimensions. Then we will move to higher dimensions and study large deviations events for the number of intersections, i.e. we will bound the probability that the number of intersections is very large. From the above we see that in high dimensions, the expected number of intersections is of constant order.

2.1 Intersections in four dimensions

As we already discussed, dimension 4 is the critical dimension for the problem of intersections. What is usually expected at the critical dimension is logarithmic corrections to mean field behaviour. The main result of this section is to prove Lawler's result on the non-intersection between a random walk and an independent two-sided random walk in $\mathbb{Z}^4$.

Theorem 2.1 (Lawler (1985))**.** *Let X^1, X^2 and X^3 be three independent simple random walks in $\mathbb{Z}^4$ starting from 0. Then as $n \to \infty$*

$$\mathbb{P}\big(X^1[1,\infty) \cap (X^2[0,n] \cup X^3[0,n]) = \emptyset, 0 \notin X^2[1,n]\big) \sim \frac{\pi^2}{8} \cdot \frac{1}{\log n}$$

The proof that we will present follows Lawler's original argument with some simplifications due to Bai and Wan [3] and Bruno Schapira [19], who generalised it to branching random walks that we will discuss in the final section.

The whole proof is based on the magic equality of Lemma 2.2 which is a consequence of the last exit decomposition formula that we will state shortly. First we need to set up some notation.

Let X be a two-sided simple random walk in $\mathbb{Z}^4$, i.e. $(X_n)_{n\geq 0}$ and $(X_{-n})_{n\geq 0}$ are two independent random walks started from 0. Let $\widetilde{X}$ be an independent simple random walk in $\mathbb{Z}^4$ also started from 0. For every integers $a < b$ we write $\mathcal{R}[a,b] = \{X_a, \ldots, X_b\}$ and for $a, b \in \mathbb{N}$ we set $\widetilde{\mathcal{R}}_n =$

$\{\widetilde{X}_a, \ldots, \widetilde{X}_b\}$ for the ranges of the two walks during the time interval $[a,b]$. Let ξ_n^ℓ and ξ_n^r be two independent geometric random variables of parameter $1/n$ each ($\mathbb{P}(\xi_n^\ell = j) = 1/n \cdot (1-1/n)^j$ for all j), also independent of the walks. Finally we define

$$\begin{aligned} \mathcal{A}_n &= \{\widetilde{\mathcal{R}}[1,\infty) \cap \mathcal{R}[-\xi_n^\ell, \xi_n^r] = \emptyset\} \\ e_n &= \mathbf{1}(0 \notin \mathcal{R}[1, \xi_n^r]) \\ G_n &= \sum_{-\xi_n^\ell \le k \le \xi_n^r} g(X_k). \end{aligned}$$

Lemma 2.2. *With the above definitions we have*

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot e_n \cdot G_n] = 1.$$

In Lemma 2.4 below we will show that G_n is concentrated around its mean (which is of order $\log n$). Hence, if we could just pull it out of the expectation above, we would get exactly the statement of the theorem. The proof will then proceed by showing that this is actually correct up to smaller order terms.

Before giving the proof of the magic formula (Lemma 2.2) we state and prove the last exit decomposition formula which is an easy consequence of the Markov property. This result will be used repeatedly throughout these notes.

Lemma 2.3 (Last exit decomposition formula)**.** *Let $d \ge 3$ and let $A \subseteq \mathbb{Z}^d$ be a finite set. Then for all $x \in \mathbb{Z}^d$ we have*

$$\mathbb{P}_x(H_A < \infty) = \sum_{y \in A} g(x,y) \mathbb{P}_y\left(\widetilde{H}_A = \infty\right).$$

Proof. Let $L_A = \sup\{t \ge 0 : X_t \in A\}$ be the last time X visits A with the convention that $L_A = -\infty$ if the set is empty. Then by transience of the walk we get $\{H_A < \infty\} = \{0 \le L_A < \infty\}$, and hence

$$\begin{aligned} \mathbb{P}_x(H_A < \infty) = \mathbb{P}_x(0 \le L_A < \infty) = \sum_{n=0}^{\infty} \sum_{y \in A} \mathbb{P}_x(L_A = n, X_n = y) &= \sum_{n=0}^{\infty} \sum_{y \in A} \mathbb{P}_x(X_n = y)\, \mathbb{P}_y\left(\widetilde{H}_A = \infty\right) \\ &= \sum_{y \in A} g(x,y) \mathbb{P}_y\left(\widetilde{H}_A = \infty\right) \end{aligned}$$

where for the penultimate equality we used the Markov property. $\square$

Proof of Lemma 2.2. For every nearest neighbour path $(x_1, \ldots, x_m)$ we define

$$B(m, x_1, \ldots, x_m) = \{\xi_n^\ell + \xi_n^r = m,\ X_{-\xi_n^\ell + k} - X_{-\xi_n^\ell} = x_k,\ \forall\, 1 \le k \le m\},$$

and for all $0 \le j \le m$ we define

$$B(m, j, x_1, \ldots, x_m) = \{\xi_n^\ell = j,\ \xi_n^r = m - j,\ X_{-\xi_n^\ell + k} - X_{-\xi_n^\ell} = x_k,\ \forall\, 1 \le k \le m\}.$$

Using the independence of the increments of the walk and the geometric random variables we then obtain

$$\mathbb{P}(B(m, j, x_1, \ldots, x_m) \mid B(m, x_1, \ldots, x_m)) = \frac{1}{m+1}.$$

Setting $x_0 = 0$, we then have

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n \cdot G_n] \\
=&\sum_{m=0}^{\infty}\sum_{(x_1,\ldots,x_m)} \frac{\mathbb{P}(B(m,x_1,\ldots,x_m))}{m+1}\cdot \sum_{k=0}^{m}\sum_{j=0}^{m}\mathbf{1}(x_j\notin\{x_{j+1},\ldots,x_m\}) \\
&\times \mathbb{P}\Big((x_j+\widetilde{\mathcal{R}}[1,\infty))\cap\{x_0,x_1,\ldots,x_m\}=\emptyset\Big)\, g(x_j-x_k).
\end{aligned}
$$

Using the last exit decomposition formula to the set $\{x_0,\ldots,x_m\}$ and the starting point x_k we get

$$
1=\sum_{j=0}^{m}\mathbf{1}(x_j\notin\{x_{j+1},\ldots,x_m\})\times \mathbb{P}\Big((x_j+\widetilde{\mathcal{R}}[1,\infty))\cap\{x_0,x_1,\ldots,x_m\}=\emptyset\Big)\, g(x_j-x_k).
$$

Substituting this above we obtain

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n\cdot G_n]=\sum_{m=0}^{\infty}\sum_{(x_1,\ldots,x_m)}\mathbb{P}(B(m,x_1,\ldots,x_m))=1,
$$

and this concludes the proof. $\square$

Lemma 2.4. *There exists a positive constant C so that the following holds. Let X be a simple random walk on $\mathbb{Z}^4$ started from 0 and let ξ be an independent geometric random variable of mean n. Then*

$$
\mathbb{E}\left[\sum_{i=0}^{\xi} g(X_i)\right]=\frac{4}{\pi^2}\cdot\log n+O(1) \quad \textit{and} \quad \mathrm{Var}\left(\sum_{i=0}^{\xi}g(X_i)\right)\le C\log n.
$$

We defer the proof of this lemma to the end of the section and we now give the

Proof of Theorem 2.1. Lemma 2.2 states that

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n\cdot G_n]=1.
$$

We now get

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n]=\frac{1}{\mathbb{E}[G_n]}+\frac{1}{\mathbb{E}[G_n]}\cdot\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n\cdot(\mathbb{E}[G_n]-G_n)]\,. \tag{2.1}
$$

Let $\varepsilon>0$ and set

$$
B=\{|G_n-\mathbb{E}[G_n]\,|\ge\varepsilon\log n\}.
$$

Then we have

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n\cdot|\mathbb{E}[G_n]-G_n|]\le\varepsilon\log n\cdot\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n]+\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot|\mathbb{E}[G_n]-G_n|\cdot\mathbf{1}(B)]\,. \tag{2.2}
$$

Using Cauchy-Schwartz for the second term together with Lemma 2.4, we obtain

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot|\mathbb{E}[G_n]-G_n|\cdot\mathbf{1}(B)]\le\sqrt{\mathbb{P}(B)\,\mathrm{Var}(G_n)}\lesssim 1.
$$

Substituting this bound into (2.2) and then into (2.1), taking ε sufficiently small, using Lemma 2.4 and rearranging we deduce

$$
\mathbb{E}[\mathbf{1}(\mathcal{A}_n)\cdot e_n]\lesssim\frac{1}{\log n}. \tag{2.3}
$$

Claim 2.5. *We have*

$$\mathbb{P}(\mathcal{A}_n) \lesssim \frac{1}{\log n} \quad and \quad \mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap \mathcal{R}(0,\xi_n^r] = \emptyset\Big) \lesssim \frac{1}{\sqrt{\log n}}. \tag{2.4}$$

We now explain that it suffices to prove that

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot |\mathbb{E}[G_n] - G_n| \cdot \mathbf{1}(B)] \lesssim \frac{1}{(\log n)^{1/4}}. \tag{2.5}$$

Indeed, once this is established, then we get

$$\left|\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot e_n] - \frac{1}{\mathbb{E}[G_n]}\right| \leq \varepsilon \cdot \mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot e_n] + \mathcal{O}\left(\frac{1}{(\log n)^{5/4}}\right),$$

and, since this holds for any $\varepsilon > 0$ and $\mathbb{E}[G_n] \sim 8/\pi^2 \log n$ by Lemma 2.4, this concludes the proof in the case where we run the two-sided walk up to two geometric times. To pass to the fixed n case, one needs to use that $\mathbb{P}\big(n/(\log n)^2 \leq \xi_n^r \leq n(\log n)^2\big) = 1 - (\log n)^{-2}$ and similarly for ξ_n^ℓ. So we now turn to prove (2.5). By the Cauchy-Schwartz inequality we obtain

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot |\mathbb{E}[G_n] - G_n| \cdot \mathbf{1}(B)] \leq \sqrt{\mathbb{P}(\mathcal{A}_n \cap B) \cdot \mathbb{E}[(\mathbb{E}[G_n] - G_n)^2]} \leq \sqrt{\mathbb{P}(\mathcal{A}_n \cap B) \cdot \log n},$$

where for the last inequality we used Lemma 2.4. It remains to bound the last probability appearing above. To do this we define

$$G_n^1 = \sum_{k=-\xi_n^\ell}^{0} g(X_k) \quad \text{and} \quad G_n^2 = \sum_{k=0}^{\xi_n^r} g(X_k),$$

and also two events for $i = 1, 2$

$$B_i = \{|G_n^i - \mathbb{E}\big[G_n^i\big]\,| \geq \varepsilon \log n/2\}.$$

Then it is clear that $B \subseteq B_1 \cup B_2$, and hence we deduce

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n \cap B) &\leq \mathbb{P}\Big(\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[-\xi_n^\ell, 0] = \emptyset, B_2\Big) + \mathbb{P}\Big(\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[0, \xi_n^r] = \emptyset, B_1\Big) \\ &= 2\mathbb{P}\Big(\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[-\xi_n^\ell, 0] = \emptyset\Big)\, \mathbb{P}(B_2) \lesssim \frac{1}{\sqrt{\log n}} \cdot \frac{1}{\log n}.\end{aligned}$$

Note that for the equality we used the independence between the two sides of the walk X and for the last step we used the concentration result, Lemma 2.4, together with (2.4). Altogether this gives

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot |\mathbb{E}[G_n] - G_n| \cdot \mathbf{1}(B)] \lesssim \frac{1}{(\log n)^{1/4}},$$

and this concludes the proof . $\square$

Proof of Claim 2.5. This proof follows closely [19]. Assuming $\mathbb{E}[\mathbf{1}(\mathcal{A}_n) \cdot e_n] \lesssim 1/\log n$, we want to show that

$$\mathbb{P}(\mathcal{A}_n) \lesssim \frac{1}{\log n}.$$

Let σ be the last time that $(X_n)_{n\geq 0}$ is at 0. Then we have

$$\mathbb{P}(\mathcal{A}_n) \leq \mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap (\mathcal{R}[-\xi^\ell_{\sqrt{n}}, 0] \cup \mathcal{R}[\sigma, \sigma + \xi^r_{\sqrt{n}}]) = \emptyset\Big) + \mathbb{P}\Big(\sigma + \xi^r_{\sqrt{n}} \geq \xi^r_n\Big),$$

where we took $\xi^r_{\sqrt{n}}$ to be an independent geometric random variable of parameter $1/\sqrt{n}$. The second probability appearing on the right-hand side above can be bounded as

$$\mathbb{P}\Big(\sigma + \xi^r_{\sqrt{n}} \geq \xi^r_n\Big) \leq \mathbb{P}\Big(\xi^r_n - \xi^r_{\sqrt{n}} < \sqrt{n}\Big) + \mathbb{P}(\sigma \geq \sqrt{n}).$$

Now it is easy to see that both these terms are much smaller than $1/\log n$.

To control the first probability on the right-hand side above we observe that the walk X after time σ has the same law as a walk started from 0 and conditioned on never returning to 0. Hence we get

$$\mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap (\mathcal{R}[-\xi^\ell_{\sqrt{n}}, 0] \cup \mathcal{R}[\sigma, \sigma + \xi^r_{\sqrt{n}}]) = \emptyset\Big) \leq \frac{1}{\mathbb{P}_0\Big(\widetilde{H}_0 = \infty\Big)} \mathbb{E}\Big[\mathbf{1}(\mathcal{A}_{\sqrt{n}}) \cdot e_n\Big] \lesssim \frac{1}{\log n},$$

where we used the transience of the walk. This now finishes the proof of the first claim.

We now turn to proving

$$\mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap \mathcal{R}[0, \xi^r_n]\Big) \lesssim \frac{1}{\sqrt{\log n}}.$$

By conditioning on $\widetilde{\mathcal{R}}$ we get

$$\begin{aligned}
&\mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap (\mathcal{R}[-\xi^\ell_n, 0] \cup \mathcal{R}[0, \xi^r_n])\Big) = \mathbb{E}\Big[\mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty) \cap (\mathcal{R}[-\xi^\ell_n, 0] \cup \mathcal{R}[0, \xi^r_n]) \,\Big|\, \widetilde{\mathcal{R}}[1,\infty)\Big)\Big] \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\mathcal{R}[-\xi^\ell_n, 0] \cap \widetilde{\mathcal{R}}[1,\infty) = \emptyset \,\Big|\, \widetilde{\mathcal{R}}[1,\infty)\Big) \mathbb{P}\Big(\mathcal{R}[0, \xi^r_n] \cap \widetilde{\mathcal{R}}[1,\infty) = \emptyset \,\Big|\, \widetilde{\mathcal{R}}[1,\infty)\Big)\Big] \\
&= \mathbb{E}\Big[\Big(\mathbb{P}\Big(\mathcal{R}[-\xi^\ell_n, 0] \cap \widetilde{\mathcal{R}}[1,\infty) = \emptyset \,\Big|\, \widetilde{\mathcal{R}}[1,\infty)\Big)\Big)^2\Big] \geq \Big(\mathbb{P}\Big(\mathcal{R}[-\xi^\ell_n, 0] \cap \widetilde{\mathcal{R}}[1,\infty) = \emptyset\Big)\Big)^2.
\end{aligned}$$

For the second equality we used the independence of the positive and negative parts of the walk and for the last inequality we used Jensen's inequality. Combining this with the first statement completes the proof. $\square$

Proof of Lemma 2.4. Using the local CLT, it is a direct calculation to check that as $n \to \infty$

$$\mathbb{E}\left[\sum_{i=0}^n g(X_i)\right] = \frac{4}{\pi^2} \cdot \log n + O(1).$$

It is straightforward to see that replacing n by a geometric random variable of parameter $1/n$ gives exactly the same asymptotics. It remains to estimate the variance. Note that if instead of the walk we were considering a Brownian motion, then we could divide this sum between the first hitting times of balls of radii 2^i for $i = 0, \ldots, \log n/2$ and we would get a sum of independent terms. With the walk one can carry through such an argument too, but there are the lattice effects that have to be taken care of. So as in Lawler's proof we simply estimate the variance using the local CLT. For this we have

$$\mathrm{Var}\left(\sum_{i=0}^n g(X_i)\right) = \sum_{i=0}^n \mathrm{Var}(g(X_i)) + \sum_{i\neq j} \mathrm{Cov}(g(X_i), g(X_j)).$$

It remains to estimate $\mathbb{E}[g(X_i)g(X_j)]$. This can be done employing the local CLT and for the details we refer the reader to [10]. $\square$

2.2 Capacity

Let $d \geq 3$. Let A be a finite subset of $\mathbb{Z}^d$. The capacity of A is defined as the sum of escape probabilities from A, i.e.

$$\mathrm{Cap}(A) = \sum_{x \in A} \mathbb{P}_x\left(\widetilde{H}_A = \infty\right).$$

We define the equilibrium measure of A to be given by

$$e_A(x) = \mathbb{P}_x\left(\widetilde{H}_A = \infty\right) \cdot \mathbf{1}(x \in A).$$

Exercise 2.6. Let $d \geq 3$ and let $A \subseteq \mathbb{Z}^d$ be a finite set. For all n we let $\mathcal{R}_n = \{X_0, \dots, X_n\}$ be the range of a simple random walk X in $\mathbb{Z}^d$. Explain why the following limit exists

$$\lim_{n \to \infty} \frac{|\mathcal{R}_n + A|}{n}$$

and identify its value. (Note that $\mathcal{R}_n + A$ denotes the Minkowski sum of $\mathcal{R}_n$ and A.)

Corollary 2.7. *Let $A \subseteq \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$. Then*

$$\mathrm{Cap}(A) = \lim_{\|x\| \to \infty} \frac{\mathbb{P}_x(H_A < \infty)}{g(x)}.$$

Proof. Recall the last exit decomposition formula

$$\mathbb{P}_x(H_A < \infty) = \sum_{y \in A} g(x,y) \mathbb{P}_y\left(\widetilde{H}_A = \infty\right).$$

Dividing both sides of this equality by $g(x)$ we get

$$\frac{\mathbb{P}_x(H_A < \infty)}{g(x)} = \sum_{y \in A} \frac{g(x,y)}{g(x)} \mathbb{P}_x\left(\widetilde{H}_A = \infty\right).$$

Since A is a finite set, using Theorem 1.7 we get

$$\frac{g(x,y)}{g(x)} \to 1 \text{ as } \|x\| \to \infty.$$

Therefore, we conclude

$$\lim_{\|x\| \to \infty} \frac{\mathbb{P}_x(H_A < \infty)}{g(x)} = \sum_{y \in A} \mathbb{P}_x\left(\widetilde{H}_A = \infty\right) = \mathrm{Cap}(A)$$

and this finishes the proof. $\square$

Exercise 2.8. Let $A, B \subseteq \mathbb{Z}^d$ be finite sets. Show that

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B) - \mathrm{Cap}(A \cap B).$$

Exercise 2.9. Let $r > 0$. Show that

$$\mathrm{Cap}(B(0,r)) \asymp r^{d-2}.$$

Exercise 2.10. Let $x, y \in \mathbb{Z}^d$. Show that

$$\mathrm{Cap}(\{0\}) = \frac{1}{g(0)} \quad \text{and} \quad \mathrm{Cap}(\{x,y\}) = \frac{2}{g(0)+g(x-y)}.$$

Remark 2.11. From the definition of capacity we see that it is intimately related to the question of intersection of a random walk with a set. If we replace the deterministic set A by a random set, then the question of capacity reduces to the question of whether a random walk intersects that independent random set.

Exercise 2.12. Let X be a simple random walk in $\mathbb{Z}^4$ and let $\mathcal{R}_n = \{X_0, \dots, X_n\}$ be its range up to time n. Using Theorem 2.1 show that

$$\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \sim \frac{\pi^2}{8} \cdot \frac{n}{\log n}.$$

Theorem 2.13. *Let $d \geq 3$ and let $A \subseteq \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$. Then*

$$\frac{1}{\mathrm{Cap}(A)} = \inf\left\{ \sum_{x,y\in A} g(x,y)\mu(x)\mu(y) : \mu \text{ probability measure on } A \right\}.$$

Proof. First of all using the last exit decomposition formula gives that with

$$\mu(x) = \frac{e_A(x)}{\mathrm{Cap}(A)},$$

we get $\sum_{x,y\in A} g(x,y)\mu(x)\mu(y) = \mathrm{Cap}(A)$. So it suffices to show that for any other probability measure μ supported on A we have

$$\sum_{x,y\in A} g(x,y)\mu(x)\mu(y) \geq \frac{1}{\mathrm{Cap}(A)}. \tag{2.6}$$

To prove this we define an inner product between any two probability measures μ and ν supported on A as follows

$$\langle \mu, \nu \rangle = \sum_{x,y\in A} \mu(x) g(x,y) \nu(y).$$

Then taking $\nu = e_A/\mathrm{Cap}(A)$, the normalised equilibrium measure, and for any μ we get using again the last exit decomposition formula

$$\langle \mu, \nu \rangle = \frac{1}{\mathrm{Cap}(A)}.$$

Now by the Cauchy-Schwartz inequality we obtain

$$\frac{1}{\mathrm{Cap}(A)} = \langle \mu, \nu \rangle \leq \sqrt{\langle \mu, \mu \rangle \langle \nu, \nu \rangle} = \sqrt{\langle \mu, \mu \rangle} \cdot \frac{1}{\sqrt{\mathrm{Cap}(A)}}.$$

Rearranging proves (2.6). $\square$

Exercise 2.14. The goal of this exercise is to show that there exists a universal constant $c > 0$ so that for any finite subset A of $\mathbb{Z}^d$ we have

$$\mathrm{Cap}(A) \geq c \cdot |A|^{1-2/d}. \tag{2.7}$$

1. Show that there exists a positive constant C so that for every $x \in A$

$$\sum_{y\in A} g(x,y) \le C|A|^{2/d}.$$

2. Taking $\mu = 1/|A|$ in the variational characterisation of capacity and using the above bound prove (2.7).

The following lemma gives yet another equivalent definition of capacity. Its usefulness will be apparent in Lemma 2.16, where the walk is required to spend a certain amount of time at each site of a set A.

Lemma 2.15. *Let $d \ge 3$ and let A be a finite subset of $\mathbb{Z}^d$. Then the capacity of A satisfies*

$$\mathrm{Cap}(A) = \sup\left\{ \sum_{x\in A} \varphi(x) : \varphi : A \to \mathbb{R}_+ \quad \textit{and} \quad \sum_{y\in A} g(x,y)\varphi(y) \le 1, \ \forall\, x \right\}.$$

Proof. First of all we see that taking $\varphi(x) = e_A(x)$ for $x \in A$ satisfies

$$\sum_{y\in A} g(x,y)\varphi(y) = \mathbb{P}_x(H_A < \infty) \le 1$$

by the last-exit decomposition formula. Moreover,

$$\sum_{x\in A} \varphi(x) = \mathrm{Cap}(A).$$

Hence, it remains to show that for any function $\varphi : A \to \mathbb{R}_+$ with $\sum_{y\in A} g(x,y)\varphi(y) \le 1$ for all x, we have that

$$\sum_{x\in A} \varphi(x) \le \mathrm{Cap}(A).$$

Now observe that using the assumption that $\sum_{y\in A} g(x,y)\varphi(y) \le 1$ for all x we have

$$\sum_{x\in A} e_A(x) \cdot \sum_{y\in A} g(x,y)\varphi(y) \le \sum_{x\in A} e_A(x).$$

By the last exit decomposition formula we also obtain

$$\sum_{x\in A} e_A(x) \cdot \sum_{y\in A} g(x,y)\varphi(y) = \sum_{y\in A} \varphi(y) \sum_{x\in A} g(x,y) e_A(x) = \sum_{y\in A} \varphi(y)\mathbb{P}_y(H_A < \infty) = \sum_{y\in A} \varphi(y).$$

Combining this with the above shows that

$$\sum_{y\in A} \varphi(y) \le \mathrm{Cap}(A)$$

and this completes the proof. $\square$

For a simple random walk X in $\mathbb{Z}^d$ with $d \ge 3$ we write $\ell(x) = \sum_{i=0}^{\infty} \mathbf{1}(X_i = x)$, for $x \in \mathbb{Z}^d$, to denote the local time at x.

Lemma 2.16. *Let A be a finite subset of $\mathbb{Z}^d$ and let $t > 0$. Then*

$$\mathbb{P}(\ell(x) \geq t, \ \forall \ x \in A) \leq 2\exp(-t \cdot \mathrm{Cap}(A)/2).$$

Remark 2.17. We write $f * g$ to denote the convolution of f and g, i.e.

$$f * g(x) = \sum_y f(x-y)g(y).$$

Lemma 2.18. *Let φ be a function satisfying $\|g * \varphi\|_\infty \leq 1$. Then for all $x_0 \in \mathbb{Z}^d$ and all $\theta \in (0,1)$ we have*

$$\mathbb{E}_{x_0}\left[\exp\left(\theta \cdot \sum_x \varphi(x)\ell(x)\right)\right] \leq \frac{1}{1-\theta}.$$

Proof. First of all we notice that we can write this quantity as

$$\sum_x \varphi(x)\ell(x) = \sum_x \varphi(x) \sum_{k=0}^{\infty} \mathbb{1}(X_k = x) = \sum_{k=0}^{\infty} \varphi(X_k).$$

We now upper bounding the n-th moment of $\sum_x \varphi(x)\ell(x)$. For this we have

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\left(\sum_{k=0}^{\infty}\varphi(X_k)\right)^n\right] = \mathbb{E}_{x_0}\left[\sum_{k_1,\dots,k_n} \varphi(X_{k_1})\cdots\varphi(X_{k_n})\right] \leq n! \sum_{k_1 \leq \dots \leq k_n} \mathbb{E}_{x_0}[\varphi(X_{k_1})\cdots\varphi(X_{k_n})] \\
&= n! \sum_{k_1 \leq \dots \leq k_n} \sum_{x_1,\dots,x_n} \mathbb{P}_{x_0}(X_{k_1} = x_1, \dots, X_{k_n} = x_n) \prod_{i=1}^n \varphi(x_i) \\
&\leq n! \sum_{k_1 \leq \dots \leq k_n} \sum_{x_1,\dots,x_n} P^{k_1}(x_0,x_1) \cdot P^{k_2-k_1}(x_1,x_2) \cdots P^{k_n-k_{n-1}}(x_{n-1},x_n) \prod_{i=1}^n \varphi(x_i) \\
&= n! \sum_{x_1,\dots,x_n} g(x_0,x_1)\cdots g(x_{n-1},x_n) \prod_{i=1}^n \varphi(x_i) \leq n!,
\end{aligned}$$

where in the last step we used the assumption on the function φ. So we now deduce

$$\mathbb{E}_{x_0}\left[\exp\left(\theta \sum_x \varphi(x)\ell(x)\right)\right] = \sum_{n=0}^{\infty} \frac{\mathbb{E}_{x_0}[(\sum_x \varphi(x)\ell(x))^n]}{n!} \cdot \theta^n \leq \frac{1}{1-\theta}$$

and this concludes the proof. □

Proof of Lemma 2.16. Let $\varphi : A \to \mathbb{R}_+$ be a function satisfying $\|g * \varphi\|_\infty \leq 1$. It follows that

$$\{\ell(x) \geq t, \ \forall \ x \in A\} \subseteq \left\{\sum_{x \in A} \ell(x)\varphi(x) \geq t \cdot \sum_{x \in A} \varphi(x)\right\}.$$

By the exponential Chebyshev inequality we now deduce for any $\theta \in (0,1)$

$$\mathbb{P}(\ell(x) \geq t, \ \forall \ x \in A) \leq \mathbb{P}\left(\sum_{x \in A} \varphi(x)\ell(x) \geq t \sum_{x \in A} \varphi(x)\right)$$

$$\leq \exp\left(-\theta \cdot t \cdot \sum_{x\in A} \varphi(x)\right) \cdot \mathbb{E}\left[\exp\left(\theta \cdot \sum_{x} \varphi(x)\ell(x)\right)\right] \leq \frac{1}{1-\theta} \cdot \exp\left(-\theta \cdot t \cdot \sum_{x\in A} \varphi(x)\right).$$

Taking now $\theta = 1/2$, optimising over all functions $\varphi : A \to \mathbb{R}_+$ with $\|g * \varphi\|_\infty \leq 1$ and using Lemma 2.15 shows that

$$\mathbb{P}(\ell(x) \geq t,\ \forall\, x \in A) \leq 2 \cdot \exp(-t \cdot \mathrm{Cap}(A)/2)$$

and this finishes the proof. □

For a set $A \subseteq \mathbb{Z}^d$ we write $\ell(A)$ for the total time spent in A by a simple random walk X, i.e.

$$\ell(A) = \sum_{x\in A} \ell(x).$$

We also write for $x \in \mathbb{Z}^d$

$$g(x, A) := \sum_{y\in A} g(x, y).$$

Lemma 2.19. *Let $d \geq 3$ and X a simple random walk on $\mathbb{Z}^d$. There exist positive constants c and C so that if A is a finite subset of $\mathbb{Z}^d$, then we have*

$$\mathbb{P}(\ell(A) \geq t) \leq C \exp\left(-ct/ \sup_{x\in\mathbb{Z}^d} g(x, A)\right).$$

Proof. Let $\varphi(x) = 1/\sup_{x\in\mathbb{Z}^d} g(x, A)$ for all $x \in A$. Then we have

$$g * \varphi(x) = \sum_{y\in A} g(x, y)\varphi(y) \leq 1.$$

Thus we can apply Lemma 2.18 to obtain for $\theta \in (0, 1)$ that

$$\mathbb{E}\left[\exp\left(\theta \cdot \sum_{x} \varphi(x)\ell(x)\right)\right] \leq \frac{1}{1-\theta}.$$

It is immediate to see that

$$\{\ell(A) \geq t\} \subseteq \left\{\sum_{x\in A} \varphi(x)\ell(x) \geq t/ \sup_{x\in\mathbb{Z}^d} g(x, A)\right\}.$$

Applying exponential Chernoff again we deduce

$$\mathbb{P}\left(\sum_{x\in A} \varphi(x)\ell(x) \geq t/ \sup_{x\in\mathbb{Z}^d} g(x, A)\right) \lesssim \exp\left(-ct/ \sup_{x\in\mathbb{Z}^d} g(x, A)\right)$$

and this completes the proof. □

Remark 2.20. Recall from Exercise 2.14 that there exists a universal constant C so that for all sets A

$$\sup_{x\in\mathbb{Z}^d} g(x, A) \leq C|A|^{2/d}.$$

Plugging this bound into the bound in Lemma 2.19 shows that

$$\mathbb{P}(\ell(A) \geq t) \lesssim \exp(-ct/|A|^{2/d}).$$

2.3 Intersections in higher dimensions

The question on large deviations on intersections of two independent random walks in dimensions $d \geq 5$ was first studied in 1994 by Khanin, Mazel, Shloshman and Sinai [9]. They proved that for all $\varepsilon > 0$ and all t sufficiently large

$$\exp(-t^{1-2/d+\varepsilon}) \leq \mathbb{P}\Big(|\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}_\infty| > t\Big) \leq \exp(-t^{1-2/d-\varepsilon}). \tag{2.8}$$

In 2004, van den Berg, Bolthausen and den Hollander [23] showed that there exists a non-negative rate function $\mathcal{I}$ such that for all $b > 0$

$$\lim_{t\to\infty} \frac{1}{t^{1-2/d}} \log \mathbb{P}\Big(|\mathcal{R}_{\lfloor bt \rfloor} \cap \widetilde{\mathcal{R}}_{\lfloor bt \rfloor}| > t\Big) = -\mathcal{I}(b).$$

(In fact, they established it for Wiener sausages, and it was later adapted to the discrete setup by Phetdrapat [16] in his PhD thesis.)

In 2020, Asselah and Schapira [2] finally managed to settle this open question by proving a large deviations principle for the infinite time horizon.

Theorem 2.21 (Asselah and Schapira [2])**.** *For $d \geq 5$, the following limit exists and is positive*

$$\mathcal{I}_\infty = \lim_{b\to\infty} \mathcal{I}(b) = \lim_{t\to\infty} -\frac{1}{t^{1-2/d}} \log \mathbb{P}\Big(|\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}_\infty| > t\Big).$$

In these notes we are going to establish the following result of Asselah and Schapira, which removes the power ε from (2.8).

Theorem 2.22 (Asselah and Schapira [2])**.** *Let $d \geq 5$ and let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be two independent ranges. There exist positive constants c_1 and c_2 so that for all $t > 0$*

$$e^{-c_2 t^{1-2/d}} \leq \mathbb{P}\Big(\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}_\infty| > t\Big) \leq e^{-c_1 t^{1-2/d}}.$$

Moreover, Asselah and Schapira are able to identify the strategy for the two walks in order to achieve a large intersection. In particular, they show that given that the size of the intersection is larger than t, a fraction close to t of them happen in a ball of radius $t^{1/d}$.

They first prove a weaker result, namely that there exists a finite number of balls of radius $t^{1/d}$, where most of the intersections happen. To reduce to a single box, they needed to appeal to the large deviations result for the finite time horizon problem.

2.3.1 Lower bound

We start by proving the lower bound of Theorem 2.22. This is the easier direction of this problem as it entails finding a specific strategy for both walks to follow in order to achieve the required event.

The main ingredient of the proof is the following result which gives a lower bound on the probability that a walk visits a fraction of a set.

Proposition 2.23. *Let X be a simple random walk in $\mathbb{Z}^d$ with $d \geq 3$ and let $\mathcal{R}_\infty = X[0,\infty)$ denote its range. There exist positive constants ρ, κ and C so that for all $r > 0$ if $\Lambda \subseteq B(0,r)$ satisfies $|\Lambda| > C$, then*

$$\mathbb{P}(|\mathcal{R}_\infty \cap \Lambda| \geq \rho|\Lambda|) \geq \exp\left(-\kappa \cdot r^{d-2}\right).$$

We start by giving the proof the lower bound and then we proceed with the proof.

Proof of lower bound of Theorem 2.22. Let $\rho < 1$ and $r > 0$ be such that $\rho^2 |B(0,r)| = t$. We then have

$$\{|\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}_\infty| \geq t\} \supseteq \{|\mathcal{R}_\infty \cap B(0,r)| \geq \rho |B(0,r)|\} \cap \{|\widetilde{\mathcal{R}}_\infty \cap (\mathcal{R}_\infty \cap B(0,r))| \geq \rho \cdot |\mathcal{R}_\infty \cap B(0,r)|\}.$$

Using the independence between the two walks and applying Proposition 2.23 we obtain

$$\mathbb{P}\Big(|\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}_\infty| \geq t\Big) \gtrsim \exp(-\kappa \cdot r^{d-2}) = \exp(-\kappa' \cdot t^{1-2/d}),$$

where κ and κ' are positive constants. This completes the proof. □

Let us first give a high level overview of the proof of Proposition 2.23. We consider the balls $B(0,5r)$ and $B(0,10r)$. We are going to count the number of excursions the walk makes across the annulus $B(0,10r) \setminus B(0,5r)$. During each excursion, the walk has a probability $1/r^{d-2}$ of hitting a given vertex of the ball $B(0,r)$. The excursions are approximately independent, as there is enough time for the walk to mix before starting the next one. So during $K_r = K \cdot \rho \cdot r^{d-2}$ excursions, a fraction ρ of the vertices of Λ will be covered. The probability that starting from $\partial B(0,10r)$ the random walk hits $\partial B(0,5r)$ is a positive constant bounded away from 1 and 0, and hence the probability of having at least K_r excursions is of order $\exp(-cK_r)$ which is of the correct order. We now need to make this argument rigorous.

Proof of Proposition 2.23. To this end we first define the successive hitting times of $\partial B(0,5r)$ and $\partial B(0,10r)$. Set $\sigma_0 = 0$ and define recursively for $i \geq 0$

$$\begin{aligned} \tau_i &= \inf\{t \geq \sigma_i : X_t \notin B(0,10r)\} \quad \text{and} \\ \sigma_{i+1} &= \inf\{t \geq \tau_i : X_t \in \partial B(0,5r)\}. \end{aligned}$$

We let $\mathcal{N}$ be the total number of excursions the walk performs, i.e.

$$\mathcal{N} = \sup\{k \geq 0 : \sigma_k < \infty\}.$$

Using the Green's function asymptotics we get that there exists a positive constant c such that

$$\mathbb{P}(\mathcal{N} \geq k) \geq \exp(-c \cdot k). \tag{2.9}$$

Set $K_r = K \cdot \rho \cdot r^{d-2}$. Let $\mathcal{G}$ be the σ-algebra generated by the total number of excursions $\mathcal{N}$ and the entrance and exit time of these excursions, i.e.

$$\mathcal{G} = \sigma(\mathcal{N}, X_{\sigma_i}, X_{\tau_i}, i \leq \mathcal{N}).$$

We now define $\Lambda_1 = \Lambda$ and inductively for $i \geq 1$

$$\mathcal{R}^{(i)} = \{X_{\sigma_i}, \ldots, X_{\tau_i}\} \quad \text{and} \quad \Lambda_{i+1} = \Lambda \setminus (\cup_{j \leq i} \mathcal{R}^{(j)}).$$

Finally set

$$Y_i = |\mathcal{R}^{(i)} \cap \Lambda_i| \mathbf{1}(\sigma_i < \infty).$$

By conditioning on $\mathcal{G}$ we deduce

$$\mathbb{P}\left(\sum_{i=1}^{K_r} Y_i > \rho|\Lambda|,\ \mathcal{N} \geq K_r\right) = \mathbb{E}\left[\mathbf{1}(\mathcal{N} \geq K_r) \cdot \mathbb{P}\left(\sum_{i=1}^{K_r} Y_i > \rho|\Lambda| \,\middle|\, \mathcal{G}\right)\right]. \tag{2.10}$$

Let $\mathcal{H}_i = \sigma(X[0,\sigma_i])$ for every i. We now define

$$M = \sum_{i=1}^{\mathcal{N}\wedge K_r} (Y_i - \mathbb{E}[Y_i \mid \mathcal{H}_i, \mathcal{G}]).$$

Note that by the orthogonality of increments we get

$$\mathbb{E}[M \mid \mathcal{G}] = 0 \quad \text{and} \quad \mathbb{E}\left[M^2 \mid \mathcal{G}\right] \le 2 \sum_{i=1}^{\mathcal{N}\wedge K_r} \mathbb{E}\left[Y_i^2 \mid \mathcal{G}\right].$$

Exercise 2.24. Using Harnack's inequality show that for all $i \le \mathcal{N}$ we have for all $x \in B(0,r)$

$$\mathbb{P}\left(x \in \mathcal{R}^{(i)} \;\middle|\; X_{\sigma_i}, X_{\tau_i}\right) \gtrsim \mathbb{P}\left(x \in \mathcal{R}^{(i)} \;\middle|\; X_{\sigma_i}\right).$$

We also have using the asymptotics of the Green's function

$$\begin{aligned}
\mathbb{P}\left(x \in \mathcal{R}^{(i)} \;\middle|\; X_{\sigma_i} = y\right) &= \mathbb{P}_y(H_x < \infty) - \sup_{z \in \partial B(0,10r)} \mathbb{P}_z(H_x < \infty) \\
&= \frac{g(x,y)}{g(0)} - \sup_{z \in \partial B(0,10r)} \frac{g(x,z)}{g(0)} \\
&= \frac{c_d}{\|x-y\|^{d-2}} - \sup_{z \in \partial B(0,10r)} \frac{c_d}{\|x-z\|^{d-2}} + O(r^{1-d}) \asymp \frac{1}{r^{d-2}}.
\end{aligned}$$

Therefore, putting everything together we deduce that for a positive constant c

$$\mathbb{E}[Y_i \mid \mathcal{G}, \mathcal{H}_i] \ge \frac{c}{r^{d-2}} \cdot |\Lambda_i|,$$

and hence on the event $\{\mathcal{N} \ge K_r\}$ and taking $K = 4/c$ in the definition of K_r this gives

$$\sum_{i=1}^{\mathcal{N}\wedge K_r} \mathbb{E}[Y_i \mid \mathcal{G}, \mathcal{H}_i] \ge K_r \cdot \frac{c}{r^{d-2}} \cdot |\Lambda_{K_r}| = 4\rho|\Lambda_{K_r}|.$$

Since $|\Lambda_{K_r}| = |\Lambda| - \sum_{i=1}^{K_r-1} Y_i$ we get on the event $\{\mathcal{N} \ge K_r\}$ for $\rho \le 1/2$

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{K_r} Y_i \le \rho|\Lambda| \;\middle|\; \mathcal{G}\right) &= \mathbb{P}\left(\sum_{i=1}^{K_r} Y_i \le \rho|\Lambda|, |\Lambda_{K_r}| \ge |\Lambda|/2 \;\middle|\; \mathcal{G}\right) \le \mathbb{P}(|M| \ge \rho|\Lambda| \mid \mathcal{G}) \\
&\le \frac{\mathbb{E}\left[M^2 \mid \mathcal{G}\right]}{\rho^2|\Lambda|^2} \le \frac{2}{\rho^2|\Lambda|^2} \sum_{i=1}^{K_r} \mathbb{E}\left[Y_i^2 \mid \mathcal{G}\right].
\end{aligned} \tag{2.11}$$

Now it remains to bound this last sum of conditional expectations. For this we obtain

$$\begin{aligned}
\mathbb{E}\left[Y_i^2 \mid \mathcal{H}_i, \mathcal{G}\right] &= \sum_{(z,z') \in \Lambda_i \times \Lambda_i} \mathbb{P}\left(z \in \mathcal{R}^{(i)}, z' \in \mathcal{R}^{(i)} \;\middle|\; X_{\sigma_i}, X_{\tau_i}\right) \\
&\le 2 \sum_{(z,z') \in \Lambda_i \times \Lambda_i} \mathbb{P}\left(z \in \mathcal{R}^{(i)}, z' \in \mathcal{R}^{(i)}, H_z < H_{z'} \;\middle|\; X_{\sigma_i}, X_{\tau_i}\right).
\end{aligned}$$

Applying the Harnack inequality again we get that up to a positive constant this last sum is equal to

$$\sum_{(z,z') \in \Lambda_i \times \Lambda_i} \mathbb{P}_{X_{\sigma_i}}(H_z < H_{z'} < \infty) \le \sum_{(z,z') \in \Lambda \times \Lambda} \frac{1}{r^{d-2}} \cdot \frac{1}{\|z-z'\|^{d-2}+1} \lesssim \frac{1}{r^{d-2}} \cdot |\Lambda|^{1+2/d}.$$

Plugging this bound into (2.11) we see that on the event $\{\mathcal{N} \geq K_r\}$ we have

$$\mathbb{P}\left(\sum_{i=1}^{K_r} Y_i \leq \rho|\Lambda| \;\middle|\; \mathcal{G}\right) \leq \frac{2}{\rho^2|\Lambda|^2} \cdot K_r \cdot \frac{1}{r^{d-2}} \cdot |\Lambda|^{1+2/d} \leq \frac{2}{c\rho|\Lambda|^{1-2/d}} \leq \frac{1}{2},$$

by taking $|\Lambda| > C$ with C a large constant so that $c \cdot \rho \cdot C^{1-2/d} \geq 4$. Plugging this bound back into (2.10) and using also (2.9) with $k = K_r$ completes the proof. $\square$

2.3.2 Upper bound

We devote this section to the proof of the upper bound of Theorem 2.22. Here we are working in dimensions $d \geq 5$.

We first start by reducing the problem to a finite time horizon, as we show that it is very unlikely for intersections to occur at high enough times. More precisely, for every $n \geq 0$ we define

$$A_n = \sum_{i=n}^{\infty} \sum_{j=0}^{\infty} \mathbf{1}(X_i = \widetilde{X}_j).$$

Using the local CLT we then obtain

$$\mathbb{E}[A_n] = \sum_{i=n}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}\Big(X_i = \widetilde{X}_j\Big) = \sum_{k=n}^{\infty} (k+1) p_k(0,0) \asymp n^{(4-d)/2}.$$

By Markov's inequality we get

$$\mathbb{P}\Big(\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[n,\infty) \neq \emptyset\Big) \leq \mathbb{E}[A_n] \asymp n^{(4-d)/2},$$

and hence taking $n = \exp(t^{1-2/d})$ gives the desired upper bound. So we can focus now on intersections between $\widetilde{\mathcal{R}}_\infty$ and $\mathcal{R}[0,n]$ for this specific value of n.

The following proposition is the main ingredient in the proof of the upper bound.

Proposition 2.25. *There exist positive constants c and C so that if $n = \exp(t^{1-2/d})$, then*

$$\mathbb{P}\left(\sup_{x \in \mathbb{Z}^d} g(x, \mathcal{R}_n) > C t^{2/d}\right) \leq C \exp(-c t^{1-2/d}).$$

Proof of upper bound of Theorem 2.22. As we explained above it suffices to study the number of intersections between $\widetilde{\mathcal{R}}_\infty$ and $\mathcal{R}_n$. By Proposition 2.25 we obtain

$$\mathbb{P}\Big(|\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}_n| > t\Big) \leq \mathbb{P}\left(|\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}_n| > t, \sup_{x \in \mathbb{Z}^d} g(x, \mathcal{R}_n) \leq C t^{2/d}\right) + C \exp(-c t^{1-2/d}).$$

Applying Lemma 2.19 to the first probability appearing on the right-hand side above we get

$$\begin{aligned}
&\mathbb{P}\left(|\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}_n| > t, \sup_{x \in \mathbb{Z}^d} g(x, \mathcal{R}_n) \leq C t^{2/d}\right) \\
&\leq \mathbb{E}\left[\exp(-ct/\sup_{x \in \mathbb{Z}^d} g(x, \mathcal{R}_n)) \mathbf{1}(\sup_{x \in \mathbb{Z}^d} g(x, \mathcal{R}_n) \leq C t^{2/d})\right] \leq \exp(-c t^{1-2/d})
\end{aligned}$$

and this concludes the proof. $\square$

The main idea behind the proof of the upper bound of Theorem 2.22, and more precisely the proof of Proposition 2.25, is that intersections will happen in the high density regions of the range of each walk, i.e. the regions that are visited a lot by the walk. We are going to perform a multiscale analysis of the high density region and then we will bound the Green's function separately for the high and the low density regions. In particular, we will show that the Green's function for the low density regions can be bounded deterministically, while for the high density we will first prove that with high probability the sizes of these regions are not too large, and then on that we will be able to bound the Green's function, thus proving Proposition 2.25.

For $r > 0$ and $\rho \in (0, 1)$ we now let

$$\mathcal{R}_n(\rho, r) = \{x \in \mathcal{R}_n : |\mathcal{R}_n \cap B(x, r)| > \rho |B(x, r)|\},$$

i.e. the set $\mathcal{R}_n(\rho, r)$ contains the points on the range for which a fraction ρ of the ball around them is covered by the range.

The following proposition controls the size of the set of high density regions.

Proposition 2.26. *There exist positive constants C, C_0 and κ such that the following holds. For any $r \geq 1$, $n \in \mathbb{N}$ and any $\rho > 0$ satisfying*

$$\rho \cdot r^{d-2} \geq C_0 \cdot \log n, \tag{2.12}$$

then for any $L \geq 1$ we have

$$\mathbb{P}(|\mathcal{R}_n(\rho, r)| > L) \leq C \exp\left(-\kappa \cdot \rho^{2/d} \cdot L^{1-2/d}\right).$$

Claim 2.27. *There exists a positive constant C so that the following holds. Let A be any finite set and $r \geq 1$ such that*

$$|A \cap B(x, r)| \leq \rho |B(x, r)|, \ \forall \ x \in A.$$

Then for all $R \geq r$ we have for all $x \in \mathbb{Z}^d$

$$|A \cap B(x, R)| \leq C \cdot \rho \cdot |B(x, R)|$$

Proof. To see this, we start by choosing $x_1 \in A \cap B(x, R)$ and then inductively for any $k \geq 0$ choose $x_{k+1} \in A \cap B(x, R) \setminus (\cup_{j \leq k} B(x_j, r))$ until this set becomes empty. Let n be the total number of x_i's picked this way. Then the balls $B(x_i, r/2)$ for $i \leq n$ are disjoint, and hence

$$R^d \asymp |B(x, R)| \geq \sum_{i=1}^{n} |B(x_i, r/2)| = n|B(0, r/2)| \asymp n \cdot r^d,$$

and hence this gives that $n \asymp R^d/r^d$. Therefore, we obtain

$$|A \cap B(x, R)| \leq \sum_{i=1}^{n} |A \cap B(x_i, r)| \leq n \cdot \rho \cdot |B(0, r)| \leq C \cdot \rho \cdot |B(0, R)|,$$

thus establishing the claim. □

Lemma 2.28. *There exists a positive constant C so that the following holds. Let A be any finite set and $r \geq 1$ such that*

$$|A \cap B(x, r)| \leq \rho |B(x, r)|, \ \forall \ x \in A.$$

Then for any $x \in \mathbb{Z}^d$ we have

$$g(x, A \cap B(x, r)^c) \leq C \cdot \rho^{1-2/d} \cdot |A|^{2/d}.$$

Proof. Let $\mathcal{S}_k = B(x, r(k+1)) \setminus B(x, rk)$ for $k \geq 1$. Then we have using integration by parts and Claim 2.27

$$g(x, A \cap B(x,r)^c) = \sum_{k\geq 1} g(x, \mathcal{S}_k \cap A) \leq \sum_{k \geq 1} \frac{|A \cap \mathcal{S}_k|}{(kr)^{d-2}} = \sum_{k\geq 1} \frac{|A \cap B(x, r(k+1))| - |A \cap B(x, rk)|}{(kr)^{d-2}}$$
$$\asymp \frac{1}{r^{d-2}} \cdot \sum_{k \geq 1} \frac{|A \cap B(x, rk)|}{k^{d-1}} \leq \frac{1}{r^{d-2}} \cdot \sum_{k\geq 1} \frac{\min(\rho (rk)^d, |A|)}{k^{d-1}} \asymp \rho^{1-2/d} \cdot |A|^{2/d}$$

and this completes the proof. $\square$

We now give the proof of Proposition 2.25 and then proceed with the proof of Proposition 2.26.

Proof of Proposition 2.25. We define a sequence of densities $\rho_i = 2^{-i}$ and radii r_i for every $i \geq 0$ by setting

$$\rho_i \cdot r_i^{d-2} = C_0 \log n, \tag{2.13}$$

where C_0 is the constant of Proposition 2.26. We now define the sets Λ_i as the regions where the density in the balls of radius r_i is at least ρ_i for the first time at level i. First recall for all $i \geq 0$

$$\mathcal{R}_n(\rho_i, r_i) = \{x \in \mathcal{R}_n : |\mathcal{R}_n \cap B(x, r_i)| > \rho_i \cdot |B(x, r_i)|\}$$

and note that $\mathcal{R}_n(\rho_0, r_0) = \emptyset$. We now set

$$\Lambda_i = \mathcal{R}_n(\rho_i, r_i) \setminus (\cup_{j \leq i-1} \mathcal{R}_n(\rho_j, r_j)) \quad \text{and} \quad \Lambda_i^* = \mathcal{R}_n \setminus (\cup_{0 \leq j \leq i-1} \mathcal{R}_n(\rho_j, r_j)).$$

Let $\mathcal{S}_k = B(x, r_k) \setminus B(x, r_{k-1})$ for every $k \geq 1$. We decompose $g(x, \mathcal{R}_n)$ as follows

$$g(x, \mathcal{R}_n) = g(x, B(x, r_0) \cap \mathcal{R}_n) + \sum_{k=1}^{\infty} g(x, \mathcal{R}_n \cap \mathcal{S}_k).$$

For the first term on the right-hand side above we have

$$g(x, B(x, r_0) \cap \mathcal{R}_n) \leq g(x, B(x, r_0)) \lesssim r_0^2 \lesssim (\log n)^{2/(d-2)} \lesssim t^{2/d}.$$

Now for every $k \geq 1$ we have

$$g(x, \mathcal{R}_n \cap \mathcal{S}_k) = \sum_{i=1}^{k} g(x, \mathcal{S}_k \cap \Lambda_i) + g(x, \mathcal{S}_k \cap \Lambda_{k+1}^*).$$

We control the Green's function of the low density region as follows

$$g(x, \mathcal{S}_k \cap \Lambda_{k+1}^*) \lesssim \frac{|\mathcal{S}_k \cap \Lambda_{k+1}^*|}{r_{k-1}^{d-2}} \lesssim \frac{\rho_k r_k^d}{r_{k-1}^{d-2}} \lesssim \frac{\log n}{r_k^{d-4}},$$

where for the second inequality we used Claim 2.27 and for the final one we used (2.13). Thus taking the sum over all $k \geq 1$ we get

$$\sum_{k \geq 1} g(x, \mathcal{S}_k \cap \Lambda_{k+1}^*) \lesssim \frac{\log n}{r_0^{d-4}} \asymp \frac{\log n}{(\log n)^{(d-4)/(d-2)}} = (\log n)^{2/(d-2)} \lesssim t^{2/d}.$$

Therefore, it remains to treat the high density region. First of all we see that since $\mathcal{R}_n \le n+1$ we get that $\Lambda_i = \emptyset$ for all i such that $\rho_i r_i^d > n+1$, which using (2.13) gives that $i \gtrsim \log n$.

We have

$$\sum_{k=1}^{\infty}\sum_{i=1}^{k} g(x, \mathcal{S}_k \cap \Lambda_i) = \sum_{i=1}^{\infty}\sum_{k\ge i} g(x, \mathcal{S}_k \cap \Lambda_i) = \sum_{i=1}^{\infty} g(x, \Lambda_i \cap B(x, r_{i-1})^c).$$

We define the good event to be

$$\mathcal{E} = \{|\Lambda_i| \le \rho_i^{-2/(d-2)} \cdot t, \ \forall\ i \ge 1\}.$$

Applying Proposition 2.26 together with the fact that for $i \gtrsim \log n$ we have $|\Lambda_i| = \emptyset$, it follows that

$$\mathbb{P}(\mathcal{E}^c) \le \sum_{i=1}^{C\log n} \exp\left(-\kappa \cdot \rho_i^{2/d} \cdot (\rho_i^{-2/(d-2)} t)^{1-2/d}\right) \lesssim \exp\left(-\kappa \cdot t^{1-2/d}\right). \tag{2.14}$$

By definition, the set Λ_i contains all the points of the range that are not of density ρ_j for all $j < i$. Therefore, we see that on the event $\mathcal{E}$ using also Lemma 2.28 we have for all $i \ge 1$

$$g(x, \Lambda_i \cap B(x, r_{i-1})^c) \lesssim \rho_{i-1}^{1-2/d} \cdot |\Lambda_i|^{2/d} \le C\rho_{i-1}^{1-2/d} \cdot \rho_i^{-4/(d(d-2))} \cdot t^{2/d} = \rho_i^{(d-4)/(d-2)} \cdot t^{2/d}.$$

Taking the sum over all i completes the proof. $\square$

For a set A we write $B(A, r) = \cup_{x\in A} B(x, r)$.

Lemma 2.29. *There exists a positive constant c so that the following holds. Let $\mathcal{C}$ be a set of points in $\mathbb{Z}^d$ at distance at least $2r$ from each other. Then for all $t > 0$ we have*

$$\mathbb{P}(\ell(B(x, r)) \ge t, \ \forall\ x \in \mathcal{C}) \le \exp\left(-c \cdot t \cdot \mathrm{Cap}(\cup_{x\in\mathcal{C}} B(x, r))/r^d\right).$$

Proof. Let φ be the equilibrium measure of $\cup_{x\in\mathcal{C}} B(x, r)$. Define $\widetilde{\varphi}$ as follows

$$\widetilde{\varphi}(y) = \frac{c_1}{r^d} \sum_{z\in B(x,r)} \varphi(z), \ \forall\, y \in B(x, r),$$

where c_1 is a positive constant to be determined in order to make $g * \widetilde{\varphi} \le 1$. Let $x_0 \in \mathbb{Z}^d$. We set $A(x_0) = \{x \in \mathcal{C} : \|x - x_0\| \ge 2r\}$. We then have

$$\sum_{x\in\mathcal{C}}\sum_{y\in B(x,r)} g(x_0, y)\widetilde{\varphi}(y) = \frac{c_1}{r^d} \cdot \sum_{x\in\mathcal{C}}\sum_{y\in B(x,r)} g(x_0, y) \sum_{z\in B(x,r)} \varphi(z).$$

We split the sum over $x \in A(x_0)$ and the complement. For $x \notin A(x_0)$, we then get that $g(x_0, y) \lesssim g(x_0, z)$ for any other $z \in \partial B(x, r)$. So we obtain

$$\sum_{x\in A(x_0)}\sum_{y\in B(x,r)} g(x_0, y) \sum_{z\in B(x,r)} \varphi(z) \lesssim r^d \cdot \sum_{x\in A(x_0)}\sum_{z\in B(x,r)} g(x_0, z)\varphi(z) \lesssim r^d$$

by the last exit decomposition formula (recall $g * \varphi \le 1$). For the sum over $A(x_0)^c$ we get

$$\sum_{x\in A(x_0)^c}\sum_{y\in B(x,r)} g(x_0, y) \sum_{z\in B(x,r)} \varphi(z) \le \mathrm{Cap}(B(0, r)) \cdot \sum_{z\in B(x_0,3r)} g(x_0, z) \lesssim r^{d-2} \cdot r^2 = r^d.$$

So there exists $c_1 > 0$ so that $g * \widetilde{\varphi} \le 1$. Notice that

$$\sum_{x\in\mathcal{C}} \widetilde{\varphi}(x) = \sum_{x\in\mathcal{C}} \sum_{y\in B(x,r)} \frac{c_1}{r^d} \cdot \varphi(y) = \frac{c_1}{r^d} \cdot \sum_{z\in \cup_{x\in\mathcal{C}} B(x,r)} \varphi(z) = \frac{c_1}{r^d} \cdot \mathrm{Cap}(\cup_{x\in\mathcal{C}} B(x,r)).$$

Applying Lemma 2.18 and exponential Chernoff we finally deduce

$$\mathbb{P}(\ell(B(x,r)) \ge t, \ \forall\, x \in \mathcal{C}) \le \mathbb{P}\left(\sum_{x\in\mathcal{C}} \widetilde{\varphi}(x)\ell(B(x,r)) \ge t \sum_{x\in\mathcal{C}} \widetilde{\varphi}(x)\right) \lesssim \exp(-c \cdot t \cdot \mathrm{Cap}(\cup_{x\in\mathcal{C}} B(x,r))/r^d)$$

and this concludes the proof. □

A final step towards proving Proposition 2.26 is the following bound on the sizes of regions with controlled density from above and below.

For $r > 0$ and $\rho \in (0,1)$ we define

$$\mathcal{R}_n^*(\rho, r) = \{x \in \mathcal{R}_n : \rho|B(x,r)| < |\mathcal{R}_n \cap B(x,r)| \le 2\rho|B(x,r)|\}.$$

These are the points of the range that the balls of radius r around them are visited a lot by the walk. The important step in the proof of the theorem is the following lemma on large deviations of the size of this set.

Lemma 2.30. *There exist positive constants C, C_0 and κ so that for all $r \ge 1$, $n \in \mathbb{N}$ and $\rho > 0$ satisfying*

$$\rho \cdot r^{d-2} \ge C_0 \cdot \log n,$$

we have for all $L \ge 1$

$$\mathbb{P}(|\mathcal{R}_n^*(\rho, r)| > L) \le C \exp\left(-\kappa \rho^{2/d} \cdot L^{1-2/d}\right).$$

Proof. Let N be the number of points in $\mathcal{R}_n^*(\rho, r)$ that are at distance at least $2r$ from each other. We start by showing that on the event $\{|\mathcal{R}_n^*(\rho, r)| > L\}$, we must have $N \ge \lfloor L/(2C\rho|B(0,2r)|)\rfloor =: n_0$, where C is a positive constant to be determined. Indeed, first pick $x_1 \in \mathcal{R}_n^*(\rho, r)$. Once we have picked $x_1, \dots, x_n$ we pick x_{n+1} from the set $\mathcal{R}_n^*(\rho, r) \setminus (\cup_{j\le n} B(x_j, 2r))$. Then using Claim 2.27 we get

$$|\mathcal{R}_n^*(\rho, r) \cap (\cup_{i=1}^N B(x_i, 2r))| \le \sum_{i=1}^N |\mathcal{R}_n \cap B(x_i, 2r)| \le C \cdot 2\rho \cdot |B(0,2r)| \cdot N,$$

where C is a positive constant. So we see that taking N as above this upper bound is smaller than $L/2$. This shows that

$$\{|\mathcal{R}_n^*(\rho, r)| > L\} \subseteq \{\exists\, \mathcal{C}\ 2r\text{-separated with } \ |\mathcal{C}| \ge n_0 \ \text{ and } \ |\mathcal{R}_n \cap B(x,r)| \ge \rho|B(x,r)|, \forall\, x \in \mathcal{C}\}$$

The total number of possible sets $\mathcal{C}$ with $|\mathcal{C}| = \ell$ is upper bounded by $(2n)^{d\cdot\ell}$. Using this, Lemma 2.29 and (2.7) we get

$$\mathbb{P}(|\mathcal{R}_n^*(\rho, r)| > L) \le \sum_{\ell \ge n_0} (2n)^{d\cdot\ell} \cdot \exp(-\kappa \cdot \rho \cdot (r^d \ell)^{1-2/d}) = \sum_{\ell\ge n_0} \exp(d \cdot \ell \cdot \log(2n) - \kappa\rho \cdot (r^d\ell)^{1-2/d}).$$

We see that the entropic term that comes from counting all possible subsets dominates in the exponential above. So we would like to reduce the total number of sets $\mathcal{C}$ that we are considering in order to match the two terms appearing in the exponential above. To do this, we will use the following result that shows that every set has a subset of the same capacity up to constants and which is of the same order as its volume.

Theorem 2.31. ([1, Theorem 1.1]) *Suppose $d \geq 3$. There exists a positive constant c so that the following holds. Let A be a finite subset of $\mathbb{Z}^d$ which is $2r$ separated for $r \geq 1$, i.e. any two distinct points of A are at distance at least $2r$ apart. Then there exists a subset U of A with the property that*

$$\mathrm{Cap}(\cup_{x\in U} B(x,r)) \geq c \cdot r^{d-2} \cdot |U| \geq c^2 \cdot \mathrm{Cap}(\cup_{x\in A} B(x,r)).$$

We defer the proof of this to end of the proof of the proposition.

Let $\mathcal{C} = \{x_1, \ldots, x_{n_0}\}$. Applying the above theorem we get that there exists a subset U of $\mathcal{C}$ such that

$$|U| \cdot r^{d-2} \asymp \mathrm{Cap}(B(U,r)) \asymp \mathrm{Cap}(B(\mathcal{C},r)).$$

Using that $\mathrm{Cap}(A) \geq |A|^{1-2/d}$ we get

$$|U| \gtrsim |B(\mathcal{C},r)|^{1-2/d} \cdot r^{2-d} \gtrsim \left(\frac{L}{\rho}\right)^{1-2/d} \cdot r^{2-d}.$$

For every $\ell > 0$ there exist at most $(2n)^{d\cdot\ell}$ possible subsets of $[-n,n]^d$ of size ℓ. So by a union bound we have

$$\begin{aligned}
&\mathbb{P}(|\mathcal{R}_n(\rho,r)| > L) \\
&\leq \sum_{\ell=(L/\rho)^{1-2/d}\cdot r^{2-d}}^{\infty} \mathbb{P}\Big(\exists\, U : |U| = \ell, |U| r^{d-2} \asymp \mathrm{Cap}(B(U,r)), |\mathcal{R}_n \cap B(x,r)| \geq \rho |B(x,r)|,\ \forall\, x \in U\Big) \\
&\leq \sum_{\ell=(L/\rho)^{1-2/d}\cdot r^{2-d}}^{\infty} \exp(c \cdot \ell \cdot \log n) \cdot \exp(-\kappa \cdot \rho \cdot \ell \cdot r^{d-2}),
\end{aligned}$$

where for the final inequality we used Lemma 2.29. By taking the constant C_0 sufficiently large so that $\rho \cdot r^{d-2} \geq C_0 \cdot \log n$, we see that there exists a positive constant κ such that the sum above is upper bounded by

$$\sum_{\ell=(L/\rho)^{1-2/d}\cdot r^{2-d}}^{\infty} \exp(-\kappa \cdot \rho \cdot \ell \cdot r^{d-2}) \lesssim \exp(-\kappa \cdot \rho^{2/d} \cdot L^{1-2/d})$$

and this completes the proof. $\square$

We are now ready to give the

Proof of Proposition 2.26. Clearly we have that

$$\mathcal{R}_n(\rho,r) = \bigcup_{i\geq 0} \mathcal{R}_n^*(2^i\rho, r).$$

Let $\alpha = \sum_{i=0}^{\infty} 2^{-i/(d-2)}$. By a union bound we get

$$\begin{aligned}
\mathbb{P}(|\mathcal{R}_n(\rho,r)| > L) &\leq \sum_{i=0}^{\infty} \mathbb{P}\left(|\mathcal{R}_n^*(2^i\rho,r)| > \alpha \cdot \frac{L}{2^{i/(d-2)}}\right) \\
&\leq \sum_{i\geq 0} \exp\left(-\kappa \cdot (2^i\rho)^{2/d} \cdot \left(\frac{L}{2^{i/(d-2)}}\right)^{1-2/d}\right) \lesssim \exp\left(-\kappa' \cdot \rho^{2/d} \cdot L^{1-2/d}\right)
\end{aligned}$$

and this completes the proof. $\square$

Proof of Theorem 2.31. We first give the proof in the case where $r = 1$. We show that every $\mathcal{C}$ has a subset U satisfying

$$\mathrm{Cap}(U) \geq c \cdot |U| \geq c^2 \mathrm{Cap}(\mathcal{C}).$$

For every $x \in A$, let (X_n^x) be a collection of independent simple random walks in $\mathbb{Z}^d$ with $X_0^x = x$ for every $x \in A$. For every $x \in A$ we write $\widetilde{H}_A^x$ for the first return time to A of the walk X^x. We now define

$$\mathcal{U} = \{x \in A : \widetilde{H}_A^x = \infty\}.$$

We then immediately get that $\mathbb{E}[|\mathcal{U}|] = \mathrm{Cap}(A)$ and $\mathrm{Var}(|\mathcal{U}|) \leq \mathrm{Cap}(A)$ as $|\mathcal{U}|$ is the sum of independent Bernoulli random variables. By Chebyshev's inequality we then obtain

$$\mathbb{P}\left(|\mathcal{U}| \leq \frac{\mathbb{E}[|\mathcal{U}|]}{2}\right) \leq \frac{4}{\mathrm{Cap}(A)} \quad \text{and} \quad \mathbb{P}(|\mathcal{U}| \geq 2\mathbb{E}[|\mathcal{U}|]) \leq \frac{1}{\mathrm{Cap}(A)}.$$

Assuming that $\mathrm{Cap}(A) > 16$, since otherwise the statement holds true, we get that

$$\mathbb{P}\left(2\mathrm{Cap}(A) \geq |\mathcal{U}| \geq \frac{1}{2}\mathrm{Cap}(A)\right) \geq \frac{2}{3}.$$

It remains to show that the capacity of $\mathcal{U}$ is of the same order as the size of $\mathcal{U}$ with high enough probability. To do this we are going to use the variational characterisation of capacity. Let μ be the uniform measure on $\mathcal{U}$. We then deduce

$$\mathrm{Cap}(\mathcal{U}) \geq \frac{|\mathcal{U}|^2}{\sum_{x,y\in\mathcal{U}} g(x,y)}. \tag{2.15}$$

We next upper bound the expectation of the denominator above. By the last exit decomposition formula we have

$$\begin{aligned}\mathbb{E}\left[\sum_{x,y\in\mathcal{U}} g(x,y)\right] &\leq \sum_{x\in A} \mathbb{P}\left(\widetilde{H}_A^x = \infty\right) g(0) + \sum_{x,y\in A} \mathbb{P}\left(\widetilde{H}_A^x = \infty\right) \mathbb{P}\left(\widetilde{H}_A^y = \infty\right) g(x,y) \\ &= g(0)\mathrm{Cap}(A) + \mathrm{Cap}(A) = (g(0)+1)\mathrm{Cap}(A).\end{aligned}$$

Using Markov's inequality we get

$$\mathbb{P}\left(\sum_{x,y\in\mathcal{U}} g(x,y) \leq 4(g(0)+1)\mathrm{Cap}(A)\right) \geq \frac{3}{4}.$$

Therefore, combining all of the above we deduce

$$\mathbb{P}\left(2\mathrm{Cap}(A) \geq |\mathcal{U}| \geq \frac{1}{2}\mathrm{Cap}(A), \sum_{x,y\in\mathcal{U}} g(x,y) \leq 4(g(0)+1)\mathrm{Cap}(A)\right) \geq \frac{5}{12}.$$

By (2.15) we see that on the event appearing in the probability above we get that

$$\mathrm{Cap}(\mathcal{U}) \geq \frac{|\mathcal{U}|^2}{4(g(0)+1)\mathrm{Cap}(A)} \geq c \cdot |\mathcal{U}| \geq c^2 \cdot \mathrm{Cap}(A),$$

where c is a positive constant, and hence, this proves that

$$\mathbb{P}\big(\mathrm{Cap}(\mathcal{U}) \geq c \cdot |\mathcal{U}| \geq c^2 \cdot \mathrm{Cap}(A)\big) \geq \frac{5}{12}.$$

This concludes the proof in the case when $r = 1$.

For $r \geq 1$ we proceed by defining for every $x \in A$ an independent Bernoulli random variable Y_x with parameter

$$\frac{c}{r^{d-2}} \cdot \sum_{y \in \partial B(x,r)} \mathbb{P}_y\Big(\widetilde{H}_{B(A,r)} = \infty\Big),$$

where c is a constant to ensure that the quantity above is smaller than 1 and we write $B(A,r) = \cup_{x\in A} B(x,r)$. We next define the set $\mathcal{U}$ as

$$\mathcal{U} = \{x : Y_x = 1\}.$$

We have for the expectation and the variance

$$\mathbb{E}[|B(\mathcal{U},r)|] = |B(0,r)| \cdot \sum_{x\in A} \frac{c}{r^{d-2}} \sum_{y\in\partial B(A,r)} \mathbb{P}_y\Big(\widetilde{H}_{B(A,r)} = \infty\Big) \asymp r^2 \cdot \mathrm{Cap}(B(A,r)).$$

For the variance as above we get

$$\mathrm{Var}(|B(\mathcal{U},r)|) \leq |B(0,r)| \cdot \mathbb{E}[|B(\mathcal{U},r)|]\,.$$

So with Chebyshev as above we get

$$\mathbb{P}\big(|B(\mathcal{U},r)| \asymp r^2 \cdot \mathrm{Cap}(B(A,r))\big) \geq \frac{3}{4}.$$

We finally need to control the sum of the Green's function as before. By taking the uniform measure on $\partial B(\mathcal{U},r)$ we need to control

$$\frac{1}{(|\mathcal{U}| \cdot |\partial B(0,r)|)^2} \sum_{x,x'\in\mathcal{U}} \sum_{y\in\partial B(x,r)} \sum_{y'\in\partial B(x',r)} g(y-y').$$

For $x = x'$ we get

$$\sum_{x\in\mathcal{U}} \sum_{y\in\partial B(x,r)} \sum_{y'\in\partial B(x,r)} g(y-y') \lesssim r^d \cdot |\mathcal{U}|.$$

For $x \neq x'$ we take expectation of the sum involving the Green's function and obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{x\neq x'\in\mathcal{U}} \sum_{y\in\partial B(x,r)} \sum_{y'\in\partial B(x',r)} g(y-y')\right] &\leq r^{2(d-1)} \cdot \sum_{x\neq x'\in A} \mathbb{P}(Y_x=1)\,\mathbb{P}(Y_{x'}=1)\,g(x-x') \\
&= r^d \cdot \sum_{x\neq x'\in A} \sum_{z\in\partial B(x,r)} \mathbb{P}_z\Big(\widetilde{H}_{B(A,r)} = \infty\Big)\,\mathbb{P}(Y_{x'}=1)\,g(x'-z) \\
&\lesssim r^d \cdot \sum_{x'\in A} \mathbb{P}(Y_{x'}=1) = r^d \cdot \mathbb{E}[|\mathcal{U}|]\,,
\end{aligned}$$

where in the last inequality we used the last exit decomposition formula. The proof can be completed in the same way as before using Chebyshev's inequality. $\square$

3 Random interlacements

3.1 Definition

The goal of this section is to define random interlacements on $\mathbb{Z}^d$ with $d \geq 3$. First we will use them in order to prove a very strong coupling result with simple random walk. In the next section we will use them in order to sample uniform spanning forests using the interlacements Aldous Broder algorithm introduced by Tom Hutchcroft.

We will write everything in the case of $\mathbb{Z}^d$ but one can generalise to any transient graph as well. For $n \leq m$ we define $\mathcal{W}(n,m)$ to be the set of graph homomorphisms from $\{n, n+1, \ldots, m\}$ to G that are transient, i.e. they have the property that every vertex is visited a finite number of times. We also define

$$\mathcal{W} = \bigcup (\mathcal{W}(n,m) : -\infty \leq n \leq m \leq \infty).$$

For every path $w \in \mathcal{W}(n,m)$ and a finite set $K \subseteq \mathbb{Z}^d$ we write

$$H_K(w) = \inf\{n \leq i \leq m : w(i) \in K\}$$

for the first time that w hits K and

$$L_K(w) = \sup\{n \leq i \leq m : w(i) \in K\}$$

for the last time w is in K. We write $w_K = w|_{[H_K(w), L_K(w)]}$. We write $\mathcal{W}_K(n,m)$ (resp. $\mathcal{W}_K$) for the paths in $\mathcal{W}(n,m)$ (resp. $\mathcal{W}$) that visit K.

We equip $\mathcal{W}$ with the topology generated by open sets of the form

$$\{w \in \mathcal{W} : w_K = w'_K\},$$

for K a finite subset of $\mathbb{Z}^d$ and $w' \in \mathcal{W}_K$. We also endow $\mathcal{W}$ with the Borel σ-algebra $\mathcal{B}(\mathcal{W})$ generated by this topology.

Finally we define the time shift $\theta_k : \mathcal{W} \to \mathcal{W}$ by assigning to every $w \in \mathcal{W}$ the path $\theta_k(w)(i) = w(i+k)$ for all i and with this we can now also define an equivalence relation $\sim$ by saying that $w_1 \sim w_2$ if there exists k such that $\theta_k(w_1) = w_2$. Lastly, define $\mathcal{W}^* = \mathcal{W}/\sim$ to be the quotient space and $\pi : \mathcal{W} \to \mathcal{W}^*$ for the projection mapping. We define the quotient σ-algebra $\widetilde{\mathcal{W}}^*$ on $\mathcal{W}^*$ by including every set A if and only if $\pi^{-1}(A) \in \mathcal{B}(\mathcal{W})$.

For every finite set $K \subseteq \mathbb{Z}^d$ we define a measure Q_K as follows

$$Q_K(\{w \in \mathcal{W} : w|_{(-\infty,0]} \in A, w(0) = x \text{ and } w|_{[0,\infty)} \in B\}) = \mathbb{P}_x\Big(X \in A, \widetilde{H}_K = \infty\Big)\, \mathbb{P}_x(X \in B)\,,$$

where A and B are Borel subsets of $\cup_{m \geq n \geq 0} \mathcal{W}(n,m)$ and X is a simple random walk. Note that we defined Q_K only on a π-system, but since the σ-algebra is generated by such sets, this uniquely determines Q_K.

From the definition of Q_K we see that $Q_K/\mathrm{Cap}(K)$ is a probability measure on bi-infinite trajectories that hit K at time 0 and $(X_n)_{n\geq 0}$ and $(X_{-n})_{n\geq 0}$ are independent conditionally on X_0 which is distributed according to the normalised equilibrium measure. Moreover, the backward path has the distribution of a simple random walk conditioned on avoiding K and the forward path is an unconditioned simple random walk.

Theorem 3.1 (Sznitman and Teixeira)**.** *There exists a unique σ-finite measure ν on $\mathcal{W}^*$ such that for every set $A \subseteq \mathcal{W}^*$ in the quotient σ-algebra of $\mathcal{W}^*$ and every finite $K \subseteq V$ we have*

$$\nu(A \cap \mathcal{W}_K^*) = Q_K(\pi^{-1}(A)).$$

Sketch of proof. To prove this result, we first show that such a measure is unique if it exists. To prove existence, we show that the measures Q_K are consistent, in the sense that if $K \subseteq K' \subseteq \mathbb{Z}^d$ are both finite subsets, then for any $A \subseteq \mathcal{W}_K^* \subseteq \mathcal{W}_{K'}^*$ we have

$$Q_{K'}(\pi^{-1}(A)) = Q_K(\pi^{-1}(A)).$$

Once this is established, then we can define ν by writing

$$\nu(A) = \sum_{n=1}^{\infty} Q_{E_n}(\pi^{-1}(A \cap (\mathcal{W}_{E_n}^* \setminus \mathcal{W}_{E_{n-1}}^*))),$$

where (E_n) is an increasing sequence of finite subsets of $\mathbb{Z}^d$ with $\cup E_n = \mathbb{Z}^d$.

For a full proof we refer the reader to [5, Theorem 6.2]. $\square$

Definition 3.2. The interlacement process $\mathcal{I}$ is a Poisson process on $\mathcal{W}^* \times \mathbb{R}$ and intensity measure $\nu \otimes \mathrm{Leb}$. We will take the canonical probability space for this Poisson process to be

$$\Omega = \left\{ \omega = \sum_n \delta_{(w_n, u_n)}, \text{ with } (w_n, u_n) \in \mathcal{W}^* \times \mathbb{R} \text{ and } \omega(\mathcal{W}_K^* \times [s,t]) < \infty \ \forall\ s \le t \text{ and } K \text{ finite} \right\}.$$

For a bi-infinite path w we write $\mathcal{R}(w)$ for the range of w, i.e.

$$\mathcal{R}(w) = \{w(k) : k \in \mathbb{Z}\}.$$

For $\omega \in \Omega$ given by $\omega = \sum_n \delta_{(w_n, u_n)}$, we write

$$\mathcal{I}^u(\omega) = \bigcup_{n:\ 0 \le u_n \le u} \mathcal{R}(w_n).$$

Sometimes it will also be convenient to think of $\mathcal{I}^u$ as

$$\mathcal{I}^u(\omega) = \sum_{n: 0 \le u_n \le u} \delta_{w_n}$$

In words, $\mathcal{I}^u$ is defined to be the set of trajectories that arrived between times 0 and u.

Lemma 3.3. *For every $u > 0$ and finite set $K \subseteq \mathbb{Z}^d$ we have*

$$\mathbb{P}(\mathcal{I}^u \cap K = \emptyset) = e^{-u \mathrm{Cap}(K)}.$$

Proof. By definition of the interlacement process we see that

$$\{\mathcal{I}^u \cap K = \emptyset\} = \{\mathcal{I}(\mathcal{W}_K^* \times [0,u]) = 0\}.$$

Since $\mathcal{I}$ is a Poisson process with intensity measure $\nu \otimes \mathrm{Leb}$, it follows that

$$\mathbb{P}(\mathcal{I}(\mathcal{W}_K^* \times [0,u]) = 0) = \exp(-\nu \otimes \mathrm{Leb}(\mathcal{W}_K^* \times [0,u])).$$

By definition of ν we obtain

$$\nu \otimes \text{Leb}(\mathcal{W}_K^* \times [0,u]) = u \cdot \nu(\mathcal{W}_K^*) = u \cdot Q_K(\mathcal{W}_K).$$

From the definition of Q_K we write

$$Q_K(\mathcal{W}_K) = \sum_{x \in K} \mathbb{P}_x\left(\widetilde{H}_K = \infty\right) = \text{Cap}(K)$$

and this together with the above concludes the proof. $\square$

We now describe an equivalent way to characterise the range of interlacements inside a finite set $K \subseteq \mathbb{Z}^d$.

For every $w \in \mathcal{W}$, we write w_+ for the forward part of w, i.e. $w_+ = (w(n))_{n \geq 0}$.

For $u > 0$ and K a finite subset of $\mathbb{Z}^d$, let $N^u \sim \text{Poisson}(u\text{Cap}(K))$. We start N^u independent random walks on ∂K and we sample their starting points independently according to the normalised equilibrium measure of K, i.e. the measure $e_K(\cdot)/\text{Cap}(K)$. Let $(X^i)_{i \leq N}$ be this collection of random walks. Then

$$\mathcal{I}^u \cap K \overset{d}{=} \bigcup_{i \leq N^u} (X^i[0,\infty) \cap K).$$

Indeed, this follows from the definition of the measure Q_K and standard properties of Poisson processes. More precisely, we define a mapping φ that takes every path in $\mathcal{W}_K^*$ to the path that hits K for the first time at time 0 and then only keeps the positive part of the path after hitting K. When we map every point of a Poisson process, we get a new Poisson process and the intensity is the push forward of the intensity measure by this mapping.

3.2 Coupling with random walk

Sznitman introduced the model of random interlacements in order to describe the range of the simple random walk on the discrete torus $\mathbb{Z}_n^d$ and disconnection problems. In this section we state and present a coupling result between random interlacements and random walk on the discrete torus $\mathbb{Z}_n^d$. The Poissonian structure of random interlacements as well as the exact expression for the probability of avoiding a set up to a given time make them very handy for calculations. Using the coupling, one can then transfer these estimates to random walks.

Let X be a simple random walk on $\mathbb{Z}_n^d$. For every $t > 0$ and $x \in \mathbb{Z}_n^d$ we write $\ell_x(t)$ for the local time at x up to time t, i.e.

$$\ell_x(t) = \sum_{i=0}^{t} \mathbf{1}(X_i = x).$$

We write $L_x(t)$ for the local time of random interlacements at x up to time t, i.e.

$$L_x(t) = \sum_{w \in \mathcal{I}^t} \sum_{n \in \mathbb{Z}} \mathbf{1}(w(n) = x).$$

We write $Q_r(0) = [-\lfloor (r-1)/2 \rfloor, \lfloor (r-1)/2 \rfloor]^d$ for the box of side length r around 0.

Theorem 3.4 (Cerny-Teixeira)**.** *Let X be a simple random walk on $\mathbb{Z}_n^d$ with $d \geq 3$ started according to the uniform distribution. For all $\delta > 0$, there exist positive constants c, C such that the following holds. For every $t > 0$, $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ there exists a coupling between $(\ell_x(t))_x$ and $(L_x(t(1 \pm \varepsilon)))_x$ so that*

$$\mathbb{P}\big(L_x(t(1-\varepsilon)) \leq \ell_x(t) \leq L_x(t(1+\varepsilon)) \ \forall\ x \in Q_{n(1-\delta)}(0)\big) \geq 1 - Cn^{2d}\lceil tn^{d-2}\rceil \exp\left(-c\varepsilon\sqrt{tn^{d-2}}\right).$$

I will present a different proof obtained in collaboration with Prévost and Rodriguez.

There are many applications of this coupling. One can transfer results about interlacements to results about the walk up to this extra ε sprinkling. Here we state an application to the study of the giant component of the vacant set of random walk on the torus $\mathbb{Z}_n^d$.

Consider running the random walk on $\mathbb{Z}_n^d$ up to time $u \cdot n^d$ for $u > 0$. Consider the set of points that the walk has not visited up to this point and denote it by

$$\mathcal{U}(un^d) = \{x \in \mathbb{Z}_n^d : H_x > un^d\}.$$

When $u = 0$, it is equal to $\mathbb{Z}_n^d$ and as u increases the size decreases and tends to 0. The question is whether there exists a phase transition in u for the size of $|\mathcal{U}(un^d)|$ analogous to the phase transition in the Erdös-Rényi random graph?

We will show that this is indeed the case when instead of the size of $\mathcal{U}$ we study its diameter. To this end, we write

$$\eta_n(u) = \mathbb{P}\Big(\mathrm{diam}(\mathcal{U}(un^d)) > n/4\Big)\,.$$

It turns out that there exists a phase transition of this quantity and the turning point is the critical density $u_\star(d)$ for random interlacements in $\mathbb{Z}^d$ that we define now.

We denote by $\mathcal{V}^u$ the vacant set of random interlacements at level u, i.e.

$$\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u.$$

It was proved by Sznitman [21] and Sidoravicius and Sznitman [20], but we will not do it here, that for all $d \geq 3$ there exists a threshold $u_\star(d) \in (0, \infty)$ so that for all $u < u_\star(d)$ the vacant set $\mathcal{V}^u$ almost surely contains a unique infinite connected component, while for $u > u_\star(d)$, all connected components are finite almost surely.

Theorem 3.5 (Černý-Teixeira [24]). *For all $d \geq 3$, there exists a threshold $u_\star(d)$ such that for all $u > u_\star(d)$*

$$\lim_{n\to\infty} \eta_n(u) = 0$$

and for $u < u_\star(d)$

$$\lim_{n\to\infty} \eta_n(u) = \eta(u),$$

where $\eta(u)$ is the probability that $0 \in \mathbb{Z}^d$ is in the infinite component of $\mathcal{V}^u$.

Proof. Using the coupling of RW with RI we have for $u > u_\star(d)$ and $\varepsilon > 0$ so that $u(1-\varepsilon) > u_\star(d)$

$$\eta_n(u) \leq \mathbb{P}\Big(\mathcal{U}(un^d) \cap Q_n(1-\delta)(0) \nsubseteq \mathcal{V}^{u(1-\varepsilon)} \cap Q_n(1-\delta)(0)\Big) + \mathbb{P}\Big(\mathrm{diam}(\mathcal{V}^{u(1-\varepsilon)}) > n/4\Big) \to 0$$

as $n \to \infty$ using Theorem 3.4 and the definition of $u_\star(d)$. Now let $u < u_\star(d)$. Using the continuity of η proved by Teixeira in [22], the fact that as $n \to \infty$ we have

$$\mathbb{P}(\mathrm{diam}(\mathcal{V}^u) > n/4) \to \eta(u)$$

and the coupling again completes the proof. $\square$

Structure of the proof of coupling There are three steps for the proof.

- First using the soft local time technique we couple together the excursions of random walk and random interlacements across the annulus $B(0, 3n/4) \setminus B(0, n/2)$.
- We prove a concentration result for the number of excursions of both random walk and random interlacements.
- Once we have the above two, we can couple the two ranges if we only consider numbers of excursions.

3.3 Soft local times

In this section we present the soft local times technique due to Popov and Teixeira from [17].

Let (Σ, μ) be a locally compact Polish metric space endowed with its Borel σ-algebra. In our applications later, the set Σ will be the set of nearest neighbour paths starting at the boundary of a smaller ball and run until they exit a bigger concentric ball.

Let $(X_n)_n$ be a Markov chain with values in the space Σ. The goal of soft local times is to give conditions in order to be able to find a coupling between X and another Markov process $\widetilde{X}$ so that under the coupling for every $\varepsilon > 0$

$$\mathbb{P}\Big(\{\widetilde{X}_s : s \leq n(1-\varepsilon)\} \subseteq \{X_s : s \leq n\} \subseteq \{\widetilde{X}_s : s \leq n(1+\varepsilon)\}\Big) \to 1 \text{ as } n \to \infty.$$

In other words, the ranges of $\widetilde{X}$ and X are close up to this sprinkling parameter ε.

This method was introduced by Popov and Teixeira as a way to decouple the random interlacements range in disjoint separated sets.

The idea of the soft local time technique is to sample the two Markov chains using a Poisson point process and then compare certain quantities (so-called soft local times) of the Markov chains in order to get a comparison between their ranges.

We are going to sample the Markov chain X using a Poisson point process on $\Sigma \times \mathbb{R}_+$ with intensity measure $\mu \otimes \mathrm{Leb}$. First we state and prove a proposition about Poisson processes that follows using standard properties of Poisson processes. Then we will use it to construct our coupling.

Proposition 3.6 (Popov-Teixeira [17])**.** *Let $\eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)}$ be a Poisson process on $\Sigma \times \mathbb{R}_+$ with intensity measure $\mu \otimes \mathrm{Leb}$, where Λ is a countable set and μ is a Radon measure, i.e. every compact set has finite μ-measure. Let $g : \Sigma \to [0, \infty)$ be a density with respect to μ, i.e. $\int g(z)\mu(dz) = 1$. Let $\xi = \inf_{\lambda \in \Lambda} \frac{v_\lambda}{g(z_\lambda)}$. Then there exists a unique $\widetilde{\lambda} \in \Lambda$ such that*

- *$\xi = \frac{v_{\widetilde{\lambda}}}{g(z_{\widetilde{\lambda}})}$ and ξ has the exponential distribution of parameter 1,*
- *$z_{\widetilde{\lambda}}$ has distribution $g \cdot \mu$ and is independent of ξ and*
- *$\eta' = \sum_{\lambda \neq \widetilde{\lambda}} \delta_{(z_\lambda, v_\lambda - \xi g(z_\lambda))}$ has the same law as η and is independent of ξ and $z_{\widetilde{\lambda}}$.*

Proof. Consider the point process

$$\widetilde{\eta} = \sum_{\lambda \in \Lambda} \delta_{\frac{v_\lambda}{g(z_\lambda)}}.$$

Then by standard properties of Poisson processes we see that $\widetilde{\eta}$ is a Poisson process with intensity measure with no atoms. Hence, there exists a unique $\widetilde{\lambda}$ such that $\xi = \frac{v_{\widetilde{\lambda}}}{g(z_{\widetilde{\lambda}})}$.

Let A be a Borel subset of Σ. Then we have

$$\mathbb{P}\big(z_{\widetilde{\lambda}} \in A\big) = \mathbb{P}\bigg(\inf_{\lambda}\{v_\lambda/g(z_\lambda) : z_\lambda \in A\} \le \inf_{\lambda}\{v_\lambda/g(z_\lambda) : z_\lambda \in A^c\}\bigg).$$

Now notice, that by standard properties of Poisson processes, the set of points $\{(z_\lambda, v_\lambda) : z_\lambda \in A\}$ is independent of the set of point $\{(z_\lambda, v_\lambda) : z_\lambda \in A^c\}$, since they are two disjoint sets. We now first find the distribution of each of the two infima appearing above. For this we have for any $t \ge 0$

$$\begin{aligned}\mathbb{P}\bigg(\inf_{\lambda}\{v_\lambda/g(z_\lambda) : z_\lambda \in A\} \ge t\bigg) &= \mathbb{P}(\eta(\{(z,v) : z \in A \text{ and } v < tg(z)\}) = 0) \\ &= \exp\left(-\int_A\int_{\mathbb{R}_+} \mu(dz)\mathbf{1}(v < tg(z))\, dv\right) = \exp\left(-t\int_A g(z)\mu(dz)\right).\end{aligned}$$

This shows that $\inf_\lambda\{v_\lambda/g(z_\lambda) : z_\lambda \in A\}$ has the exponential distribution with parameter $\int_A g(z)\mu(dz)$. Taking $A = \Sigma$, we get that ξ has the exponential distribution of parameter 1. So we can now obtain

$$\mathbb{P}\big(z_{\widetilde{\lambda}} \in A\big) = \frac{\int_A g(z)\mu(dz)}{\int_A g(z)\mu(dz) + \int_{A^c} g(z)\mu(dz)} = \int_A g(z)\mu(dz),$$

as g is a density with respect to μ.

Moreover, we have

$$\mathbb{P}\big(z_{\widetilde{\lambda}} \in A, \xi \ge t\big) = \mathbb{P}\bigg(t \le \inf_{\lambda}\{v_\lambda/g(z_\lambda) : z_\lambda \in A\} \le \inf_{\lambda}\{v_\lambda/g(z_\lambda) : z_\lambda \in A^c\}\bigg) = e^{-t}\cdot\int_A g(z)\,\mu(dz),$$

which shows the required independence between ξ and $z_{\widetilde{\lambda}}$.

For the last claim, we first define a new point process

$$\eta'' = \sum_{\lambda \ne \widetilde{\lambda}} \delta_{(z_\lambda, v_\lambda)}.$$

Given ξ and $z_{\widetilde{\lambda}}$, we see that η'' contains all points of η with the property that $v_\lambda > \xi g(z_\lambda)$. Therefore, conditional on ξ we get that η'' is a thinned version of η and thus given ξ it has intensity measure $\mathbf{1}(v > \xi g(z))\mu \otimes \text{Leb}$. Finally we see that η' is a transformation of η'', and hence we can conclude that given ξ and $z_{\widetilde{\lambda}}$, the process η' is a Poisson process of intensity $\mu \otimes \text{Leb}$. Since the distribution of η' conditionally on ξ and $z_{\widetilde{\lambda}}$ does not depend on them, it follows that η' is independent of ξ and $z_{\widetilde{\lambda}}$ and this completes the proof. $\square$

How do we sample a Markov chain using the Poisson process η? Suppose that X is a (possibly time-inhomogeneous) Markov chain with values in Σ, $X_0 = x_0$ and for every $n \ge 0$

$$\mathbb{P}(X_{n+1} \in dz \mid X_n) = g_{n+1}(X_n, z)\mu(dz) \ \text{ a.s.},$$

where $g_i(x,\cdot)$ for any $i > 0$ is a density with respect to μ for every $x \in \Sigma$, i.e. $\int g_i(x,z)\, d\mu(z) = 1$ for all $x \in \Sigma$.

Proposition 3.7. *Let $G_0(z) = 0$, $z_{\lambda_0} = x_0$ and for all $n \ge 0$ we define*

$$\xi_{n+1} = \inf_{\lambda \in \Lambda\setminus\{\lambda_1,\ldots,\lambda_n\}} \frac{v_\lambda - G_n(z_\lambda)}{g_{n+1}(z_{\lambda_n}, z_\lambda)} \ \text{ and } \ G_{n+1}(z) = G_n(z) + \xi_{n+1}g_{n+1}(z_{\lambda_n}, z)\ \forall\ z \in \Sigma.$$

Suppose the infimum above is reached at $(z_{\lambda_{n+1}}, v_{\lambda_{n+1}})$. Then the ξ_i's are i.i.d. Exponential random variables of parameter 1 and $(X_i)_{i\in\mathbb{N}}$ has the same law as $(z_{\lambda_i})_{i\in\mathbb{N}}$. Moreover, we have

$$\lambda \in \{\lambda_1, \ldots, \lambda_n\} \Leftrightarrow v_\lambda \le G_n(z_\lambda) = \sum_{k=1}^n \xi_k g_k(z_{\lambda_{k-1}}, z_\lambda).$$

Proof. Using Proposition 3.6 we see that for all n

$$\eta_n = \sum_{\lambda \in \Lambda \setminus \{\lambda_1, \dots, \lambda_n\}} \delta_{(z_\lambda, v_\lambda - G_n(z_\lambda))}$$

is a Poisson process independent of $\{z_{\lambda_1}, \dots, z_{\lambda_n}\}$ and $z_{\lambda_{n+1}}$ has density $g_{n+1}(z_{\lambda_n}, \cdot)$ with respect to μ conditionally on $\{z_{\lambda_1}, \dots, z_{\lambda_n}\}$. This proves that (z_{λ_i}) has the same law as X.

For the second part of the proposition, first note that if $\lambda = \lambda_1$, then the claim trivially holds. We proceed by induction. If $\lambda \in \{\lambda_1, \dots, \lambda_{n+1}\}$, then either $\lambda \in \{\lambda_1, \dots, \lambda_n\}$ or

$$v_\lambda = \xi_{n+1} g_{n+1}(z_{\lambda_n}, z_\lambda) + G_n(z_\lambda).$$

In the first case, we get by the induction hypothesis that $v_\lambda \leq G_n(z_\lambda)$ and in the second case we get $v_\lambda = G_{n+1}(z_\lambda)$. So in both cases we get $v_\lambda \leq G_{n+1}(z_\lambda)$. And, all these implications work in the opposite direction too, so the proof is complete. □

Definition 3.8. The process

$$G_n(z) = \sum_{k=1}^{n} \xi_k g_k(z_{\lambda_{k-1}}, z), \ \forall \ n \geq 0 \ \text{ and } \ z \in \Sigma$$

is called the soft local times process associated to the Markov chain X. Note that by the above construction we get that the ξ_k's are i.i.d. exponential random variables of parameter 1.

Now we can explain the usefulness of soft local times. We already saw how we can sample a Markov chain using a Poisson process η. Now we show how we can couple two Markov chains and compare their ranges by instead comparing their soft local times.

For every $i > 0$ let us fix some other densities $h_i : \Sigma \times \Sigma \to [0, \infty)$ with respect to μ, i.e. $\int h_i(x,z)\mu(dz) = 1$ for every $x \in \Sigma$. Using the previous procedure we can now sample from η the following

- $(x_i)_i = (z_{\lambda_i})_i$ with the same law as (X_i) and
- $(\widehat{x}_i)_i$ with the same law as a Markov chain with transition densities h_i.

Let G and $\widehat{G}$ be the corresponding soft local times. Then by Proposition 3.7 we deduce the following:

$$\text{If } \ \widehat{G}_{n(1-\varepsilon)}(z) \leq G_n(z) \leq \widehat{G}_{n(1+\varepsilon)}(z) \ \forall \ z \in \Sigma, \ \text{ then}$$
$$\{\widehat{x}_1, \dots, \widehat{x}_{n(1-\varepsilon)}\} \subseteq \{x_1, \dots, x_n\} \subseteq \{\widehat{x}_1, \dots, \widehat{x}_{n(1+\varepsilon)}\}.$$

Indeed, by Proposition 3.7 we get

$$\{x_1, \dots, x_n\} = \{z \in \Sigma : \ \exists \ (z,v) \in \eta \ \text{ s.t. } \ v \leq G_n(z)\} \ \text{ and}$$
$$\{\widehat{x}_1, \dots, \widehat{x}_n\} = \{z \in \Sigma : \ \exists \ (z,v) \in \eta \ \text{ s.t. } \ v \leq \widehat{G}_n(z)\},$$

and hence the above claim follows immediately.

Corollary 3.9. *Let $\varepsilon > 0$ and suppose that for all y, z we have*

$$1 - \frac{\varepsilon}{4} \leq \frac{g(y,z)}{f(z)} \leq 1 + \frac{\varepsilon}{4} \ \ \textit{and} \ \ 1 - \frac{\varepsilon}{4} \leq \frac{h(y,z)}{f(z)} \leq 1 + \frac{\varepsilon}{4}.$$

Then with exponentially large probability

$$\{\widehat{x}_1, \dots, \widehat{x}_{n(1-\varepsilon)}\} \subseteq \{x_1, \dots, x_n\} \subseteq \{\widehat{x}_1, \dots, \widehat{x}_{n(1+\varepsilon)}\}.$$

Proof. Now, using concentration of the sum of exponential random variables, we get that with exponentially large probability

$$\widehat{G}_{n(1-\varepsilon)}(z) = f(z) \cdot \sum_{i=1}^{n(1-\varepsilon)} \widehat{\xi}_i \leq n\left(1 - \frac{\varepsilon}{2}\right) f(z) \quad \text{and}$$
$$\widehat{G}_{n(1+\varepsilon)}(z) = f(z) \cdot \sum_{i=1}^{n(1+\varepsilon)} \widehat{\xi}_i \geq n\left(1 + \frac{\varepsilon}{2}\right) f(z).$$

So if one could compare the density g to the density h, in the sense that $g(y,z)$ is close to $h(z)$ for all y and z, then we would be done using the concentration of the sum of the exponential random variables (ξ_i).

□

3.3.1 RW/RI excursions

In this section we show how we can couple the excursions of the SRW with those of RI across an annulus. We start by giving some definitions and fixing some notation.

Let $B_1 = B(0, n/4)$, $B_2 = B(0, n/2)$ and $B_3 = B(0, 3n/4)$. We are going to be interested in the range of the walk inside of B_1. We will suppose that the walk starts outside of B_2.

We first define everything for a simple random walk on the torus and then we will explain the interlacements case. For a set A and a time t we write

$$L_A(t) = \sup\{s \leq t : X_s \in A\}$$

for the last time before time t that the walk is in the set A.

We set $\rho_0 = 0$ and for $k \geq 0$ we define inductively

$$\widetilde{\rho}_k = \inf\{t \geq \rho_k : X_t \in \partial B_2\} \quad \text{and}$$
$$\rho_{k+1} = \inf\{t \geq \widetilde{\rho}_k : X_t \in \partial B_3\}.$$

For every $k \geq 0$ we let

$$\overline{\rho}_k = \inf\{t \in [\widetilde{\rho}_k, \rho_{k+1}] : X_t \in \partial B_1\},$$

i.e. this is the first time during the time interval $[\widetilde{\rho}_k, \rho_{k+1}]$ that the random walk visits B_1. As usual the infimum of the empty set is taken to be $+\infty$.

We define the clothesline process to be the sequence $(\zeta_i)_{i\geq 1}$ given by

$$\zeta_i = (X_{\widetilde{\rho}_{i-1}}, X_{\rho_i}), \quad \text{for } i \geq 1.$$

We see that for every $i \geq 1$ we have $\zeta_i \in \partial B_2 \times \partial B_3$.

We now define $\mathcal{N}_{\mathrm{RW}}(t)$ to be the total number of excursions the walk performs before time t, i.e.

$$\mathcal{N}_{\mathrm{RW}}(t) = \sup\{k \geq 0 : \rho_k < t\}.$$

For each excursion we are now going to consider the part of the excursion that intersects the set B_1. To this end for every $i \geq 1$ we define

$$Z_i = \begin{cases} X[\overline{\rho}_{i-1}, L_{B_2}(\rho_i)] & \text{if } \overline{\rho}_{i-1} < \infty \\ \Theta & \text{otherwise} \end{cases},$$

where Θ is a cemetery state for the case where X does not hit B_1 during $[\widetilde{\rho}_{i-1}, \rho_i]$.

In the case of random interlacements at level u as we have already seen we can express the random interlacements inside of B_3 as the union of the ranges of a Poisson number of parameter $u\mathrm{Cap}(B_3)$ of independent random walk trajectories started independently according to the normalised equilibrium measure of B_3. We let N^u be the Poisson random variable of trajectories started in B_3 according to the normalised equilibrium measure.

Let $(X^j)_{j\le N^u}$ be these independent random walks in $\mathbb{Z}^d$. For each walk we define the sequence of times $(\widetilde{\rho}_k)_k$ and $(\rho_k)_k$ as above. Since the walks now are in $\mathbb{Z}^d$ and not on the torus $\mathbb{Z}_n^d$ anymore, each walk is going to perform a finite number of excursions. We thus set for each j

$$T^j = \sup\{k \ge 0 : \rho_k(X^j) < \infty\}$$

to be the total number of excursions that the walk X^j performs. We also define the clothesline process for the interlacements by taking first the clothesline process of the walk X^1, then that of X^2 and so on. We also define the parts of each excursion intersecting B_1 as before separately for each walk. We write

$$\mathcal{N}_{\mathrm{RI}}(u) = \sum_{j=1}^{N^u} T^j$$

for the total number of excursions of the walks of interlacements of level u intersecting B_3.

We now let $\mathcal{K}$ be the set of paths in B_3 that start from B_1 and end in ∂B_2. More precisely,

$$\mathcal{K} = \{\sigma = (\sigma_0, \dots, \sigma_\ell) : \sigma_0 \in B_1, \sigma_j \in B_3 \ \forall\ j \le \ell,\ \sigma_\ell \in \partial B_2\}.$$

Recall that Θ is the cemetery state corresponding to excursions that do not hit B_1. We set $\Sigma = \mathcal{K} \cup \{\Theta\}$ and for every $S \subseteq \Sigma$ we define

$$\mu(S) = \sum_{x\in B_1, y\in\partial B_2} \mathbb{P}_x\Big(X[0, L_{B_2}(H_{\partial B_3})] \in S \ \Big|\ X_{L_{B_2}(H_{\partial B_3})} = y\Big) + \mathbf{1}(\Theta \in S).$$

For $(u, w) \in \partial B_2 \times \partial B_3$ and $z = (z_0, \dots, z_\ell) \in \mathcal{K}$ we set

$$\begin{aligned} g_{(u,w)}(z) &= \mathbb{P}_u\Big(X_{H_{B_1}\wedge H_{\partial B_3}} = z_0, X_{L_{B_2}(H_{\partial B_3})} = z_\ell \ \Big|\ X_{H_{\partial B_3}} = w\Big) \quad \text{and} \\ g_{(u,w)}(\Theta) &= \mathbb{P}_u\Big(H_{\partial B_3} < H_{B_1} \ \Big|\ X_{H_{\partial B_3}} = w\Big). \end{aligned}$$

With these definitions we have for all $z \in \Sigma$

$$\mathbb{P}_u\Big(X[H_{B_1} \wedge H_{\partial B_3}, L_{B_2}(H_{\partial B_3})] = z \ \Big|\ X_{H_{\partial B_3}} = w\Big) = g_{(u,w)}(z)\mu(\{z\}).$$

Conditionally on the clothesline process $(\zeta_i)_{i\ge 0}$ the sequence $(Z_i)_{i\ge 0}$ is a time inhomogeneous Markov chain with transition densities g_{ζ_i} with respect to μ. Also conditionally on (ζ_i), the variables (Z_i) are independent. The transition density at the i-th step is of the form

$$(z', z) \mapsto g_{\zeta_i}(z),$$

which shows that it is a function only of the second variable.

3.3.2 Clothesline process

Let $(\zeta_i)_{i\geq 1}$ and $(\widetilde{\zeta}_i)_{i\geq 1}$ be the clothesline processes for the random walk and the random interlacements respectively.

Let A and B be two subsets of $\mathbb{Z}^d$ or $\mathbb{Z}_n^d$. We write

$$e_{A,B}(x) = \mathbb{P}_x\left(\widetilde{H}_A > H_B\right) \mathbf{1}(x \in A).$$

Let $B_1 \subseteq B_2$ be two concentric balls and consider the clothesline processes corresponding to these two balls. For the random walk, this corresponds to the successive entrance and exit times, while for interlacements it corresponds to successive entrance and exit times for each walk up until it escapes to infinity. Then we start another walk according to the normalised equilibrium measure on ∂B_2 and so on.

Lemma 3.10. *Let $B_1 \subseteq B_2$ be two concentric balls in $\mathbb{Z}^d$ (or $\mathbb{Z}_n^d$). Then both clothesline processes are Markov chains with invariant measure given by*

$$\nu(x_1, x_2) = e_{B_1,B_2}(x_1)\mathbb{P}_{x_1}\left(X_{H_{\partial B_2}} = x_2\right) \quad \text{for} \quad (x_1, x_2) \in \partial B_1 \times \partial B_2.$$

Lemma 3.11. *Let A, B be two disjoint sets. Then for all $y \in B$*

$$\mathbb{P}_{e_{A,B}}(X_{H_B} = y) = \mathbb{P}_y\left(H_A < \widetilde{H}_B < \infty\right).$$

Proof. The proof is the same for both random walk and random interlacements. All that changes is that in the finite case all the hitting times are always finite almost surely. For $y \in B$ we have

$$\begin{aligned}
&\mathbb{P}_{e_{A,B}}(X_{H_B} = y) = \sum_{x\in A} \mathbb{P}_x\left(\widetilde{H}_A > H_B\right) \mathbb{P}_x(X_{H_B} = y) \\
&= \sum_{x\in A}\sum_{n=0}^{\infty}\sum_{z\in A} \mathbb{P}_x\left(\widetilde{H}_A > H_B\right) \mathbb{P}_x(X_{H_B} = y, L_A(H_B) = n, X_n = z) \\
&= \sum_{x,z\in A}\sum_{n=0}^{\infty} \mathbb{P}_x\left(\widetilde{H}_A > H_B\right) \mathbb{P}_z\left(X_{H_B} = y, \widetilde{H}_A > H_B\right) \mathbb{P}_x(X_n = z, H_B > n) \\
&= \sum_{x,z\in A}\sum_{n=0}^{\infty} \mathbb{P}_x\left(\widetilde{H}_A > H_B\right) \mathbb{P}_y\left(X_{H_A} = z, \widetilde{H}_B > H_A\right) \mathbb{P}_z(X_n = x, H_B > n) \\
&= \sum_{x,z\in A}\sum_{n=0}^{\infty} \mathbb{P}_z(X_n = x, L_A(H_B) = n, H_B < \infty)\, \mathbb{P}_y\left(X_{H_A} = z, \widetilde{H}_B > H_A\right) \\
&= \sum_{z\in A} \mathbb{P}_z(H_B < \infty)\, \mathbb{P}_y\left(X_{H_A} = z, \widetilde{H}_B > H_A\right) = \mathbb{P}_y\left(H_A < \widetilde{H}_B < \infty\right),
\end{aligned}$$

where we used reversibility in the third equality twice. This completes the proof in both cases of $\mathbb{Z}_n^d$ and $\mathbb{Z}^d$. $\square$

Proof of Lemma 3.10. We first prove the claim for the clothesline of the random walk. For this we need to show that

$$\mathbb{P}_\nu(\zeta_1 = (y_1, y_2)) = e_{B_1,B_2}(y_1)\mathbb{P}_{y_1}\left(X_{H_{\partial B_2}} = y_2\right).$$

Abusing notation, here we write e_{B_1,B_2} for $e_{B_1,\partial B_2}$. We have

$$\begin{aligned}
&\mathbb{P}_\nu(\zeta_1=(y_1,y_2)) = \sum_{(x_1,x_2)\in\partial B_1\times\partial B_2} e_{B_1,B_2}(x_1)\mathbb{P}_{x_1}\left(X_{H_{\partial B_2}}=x_2\right)\mathbb{P}_{x_2}(\zeta_1=(y_1,y_2))\\
&=\sum_{x_2\in\partial B_2}\mathbb{P}_{e_{B_1,B_2}}\left(X_{H_{\partial B_2}}=x_2\right)\mathbb{P}_{x_2}(\zeta_1=(y_1,y_2)) = \sum_{x_2\in\partial B_2}\mathbb{P}_{x_2}\left(H_{B_1}<\widetilde{H}_{\partial B_2}\right)\mathbb{P}_{x_2}(\zeta_1=(y_1,y_2))\\
&=\sum_{x_2\in\partial B_2}\mathbb{P}_{x_2}\left(H_{B_1}<\widetilde{H}_{\partial B_2}\right)\mathbb{P}_{x_2}\left(X_{H_{B_1}}=y_1\right)\mathbb{P}_{y_1}\left(X_{H_{\partial B_2}}=y_2\right)\\
&=\mathbb{P}_{e_{B_2,B_1}}\left(X_{H_{B_1}}=y_1\right)\mathbb{P}_{y_1}\left(X_{H_{\partial B_2}}=y_2\right)=\mathbb{P}_{y_1}\left(H_{B_2}<\widetilde{H}_{B_1}\right)\mathbb{P}_{y_1}\left(X_{H_{\partial B_2}}=y_2\right)\\
&=e_{B_1,B_2}(y_1)\mathbb{P}_{y_1}\left(X_{H_{\partial B_2}}=y_2\right)=\nu(y_1,y_2).
\end{aligned}$$

This completes the proof in the case of a random walk.

The distribution of the clothesline for the RI case is given by for $(x_1,x_2)\in\partial B_1\times\partial B_2$ and $y\in\partial B_1$

$$\mathbb{P}_{(x_1,x_2)}(\zeta_1^1=y)=\mathbb{P}_{x_2}(X_{\widetilde{\rho}_0}=y)+\mathbb{P}_x(\widetilde{\rho}_0=\infty)\,\overline{e}_{B_1}(y),$$

i.e. if the trajectory escapes after hitting ∂B_2, then we sample the next clothesline point according to the normalised equilibrium measure. If the previous trajectory does not escape, then the next point is the hitting point of ∂B_2. Note that we write ζ^1 for the first coordinate of ζ.

In order to show that ν is an invariant measure, it suffices to show that for all $y\in\partial B_1$

$$\mathbb{P}_\nu(\zeta_1^1=y)=e_{B_1,B_2}(y).$$

We have

$$\begin{aligned}
\mathbb{P}_\nu(X_{\widetilde{\rho}_0}=y)&=\sum_{x\in\partial B_2}\mathbb{P}_{e_{B_1,B_2}}\left(X_{H_{\partial B_2}}=x\right)\mathbb{P}_x\left(X_{H_{B_1}}=y\right)\\
&=\sum_{x\in\partial B_2}\mathbb{P}_x\left(H_{B_1}<\widetilde{H}_{\partial B_2}<\infty\right)\mathbb{P}_x\left(X_{H_{B_1}}=y\right)=\sum_{x\in\partial B_2}e_{B_1,B_2}(x)\mathbb{P}_x\left(X_{H_{B_1}}=y\right)\\
&=\mathbb{P}_{e_{B_1,B_2}}\left(X_{H_{B_1}}=y\right)=\mathbb{P}_y\left(H_{\partial B_2}<\widetilde{H}_{B_1}<\infty\right),
\end{aligned}$$

since $H_{B_1}<\widetilde{H}_{\partial B_2}$ implies that $\widetilde{H}_{\partial B_2}<\infty$ as well. We also have

$$\begin{aligned}
\mathbb{P}_\nu(\widetilde{\rho}_0=\infty)&=\sum_{x\in\partial B_2}\mathbb{P}_{e_{B_1,B_2}}\left(X_{H_{\partial B_2}}=x\right)\mathbb{P}_x(H_{B_1}=\infty)\\
=\sum_{x\in\partial B_2}\mathbb{P}_x\left(H_{B_1}<\widetilde{H}_{\partial B_2}<\infty\right)\mathbb{P}_x(H_{B_1}=\infty)&=\sum_{x\in\partial B_2}e_{B_2,B_1}(x)\mathbb{P}_x(H_{B_1}=\infty)\\
&=\sum_{x\in\partial B_2}e_{B_2,B_1}(x)-\sum_{x\in\partial B_2}\sum_{y\in\partial B_1}e_{B_2,B_1}(x)\mathbb{P}_x\left(X_{H_{\partial B_1}}=y\right)
\end{aligned}$$

We now explain that

$$\sum_{x\in\partial B_2}e_{B_2,B_1}(x)=\sum_{x\in B_1}e_{B_1,B_2}(x).$$

Using that $H_{B_1}<\widetilde{H}_{B_2}$ implies that $\widetilde{H}_{B_2}<\infty$, we have

$$\sum_{x\in\partial B_2}e_{B_2,B_1}(x)=\sum_{x\in\partial B_2}\mathbb{P}_x\left(\widetilde{H}_{B_2}>H_{B_1}\right)=\sum_{x\in\partial B_2}\mathbb{P}_x\left(H_{B_1}<\widetilde{H}_{B_2}<\infty\right)$$

$$= \sum_{x\in\partial B_2} \mathbb{P}_{e_{B_1,B_2}}\Big(X_{H_{\partial B_2}} = x\Big) = \sum_{x\in\partial B_2}\sum_{y\in B_1} e_{B_1,B_2}(y)\mathbb{P}_y\Big(X_{H_{\partial B_2}} = x\Big) = \sum_{y\in B_1} e_{B_1,B_2}(y),$$

where the final equality follows from the fact that starting from B_1, the walk will exit B_2 in finite time almost surely. Plugging this above we deduce

$$\begin{aligned}\mathbb{P}_\nu(\widetilde{\rho}_0 = \infty) &= \sum_{x\in\partial B_2} e_{B_2,B_1}(x) - \sum_{y\in\partial B_1} \mathbb{P}_{e_{B_2,B_1}}\Big(X_{H_{\partial B_1}} = y\Big) \\ &= \sum_{x\in\partial B_2} e_{B_2,B_1}(x) - \sum_{y\in\partial B_1} \mathbb{P}_y\Big(H_{\partial B_2} < \widetilde{H}_{B_1} < \infty\Big) \\ &= \sum_{y\in\partial B_1} e_{B_1,B_2}(y) - \sum_{y\in\partial B_1} \mathbb{P}_y\Big(H_{\partial B_2} < \widetilde{H}_{B_1} < \infty\Big) = \sum_{y\in\partial B_1} \mathbb{P}_y\Big(\widetilde{H}_{B_1} = \infty\Big) = \mathrm{Cap}(B_1).\end{aligned}$$

Combining all of the above we obtain for $y \in B_1$

$$\begin{aligned}&\mathbb{P}_\nu\big(\zeta_1^1 = y\big) = \mathbb{P}_y\Big(H_{\partial B_2} < \widetilde{H}_{B_1} < \infty\Big) + \overline{e}_{B_1}(y)\cdot \mathrm{Cap}(B_1) \\ &= \mathbb{P}_y\Big(H_{\partial B_2} < \widetilde{H}_{B_1} < \infty\Big) + \mathbb{P}_y\Big(\widetilde{H}_{B_1} = \infty\Big) = \mathbb{P}_y\Big(H_{\partial B_2} < \widetilde{H}_{B_1}\Big) = e_{B_1,B_2}(y)\end{aligned}$$

and this concludes the proof. □

Exercise 3.12. The goal of this exercise is to prove that the mixing time of the clothesline process is of constant order.

For $x \in \partial B_3$ we write ν_x for the law of the first hitting point in ∂B_2. Then by Harnack we get that there exists a positive constant c so that for all $x, z \in \partial B_3$

$$c\cdot \nu_z \le \nu_x \le \frac{1}{c}\cdot \nu_z.$$

Using this describe a coupling when the two chains start from different points. Show that the probability that they don't couple by time k is of order $e^{-c_1 k}$ for some positive constant c_1.

Exercise 3.13. Let $K \subseteq K' \subseteq \mathbb{Z}^d$ be finite subsets. Prove the sweeping identity, i.e.

$$e_K(y) = \mathbb{P}_{e_{K'}}(H_K < \infty, X_{H_K} = y) \;\; \forall\, y \in K.$$

Deduce that

$$\mathbb{P}_{\widetilde{e}_{K'}}(H_K < \infty) = \frac{\mathrm{Cap}(K)}{\mathrm{Cap}(K')}.$$

3.3.3 Concentration of soft local times

Let $(\xi_i)_{i\ge1}$ and $(\widetilde{\xi}_i)_{i\ge1}$ be i.i.d. families of exponential random variables of parameter 1. Let (ζ_i) and $(\widetilde{\zeta}_i)_i$ be the clothesline processes of SRW on $\mathbb{Z}_n^d$ and random interlacements on $\mathbb{Z}^d$ respectively.

We write for $n \ge 1$

$$G_n(z) = \sum_{i=1}^n g_{\zeta_i}(z) \;\; \text{and} \;\; \widetilde{G}_n(z) = \sum_{i=1}^n g_{\widetilde{\zeta}_i}(z),$$

where N^u is a Poisson random variable of parameter $u\mathrm{Cap}(B_3)$.

Let $\zeta \sim e_{B_2,B_3}$. For every $z \in \Sigma$ we write

$$\overline{g}(z) = \mathbb{E}[g_\zeta(z)]\,.$$

Exercise 3.14. Using Harnack's inequality show that $g_{u,w}(z)$ is independent of u and w up to constants, i.e. that there exist c_1 and c_2 positive constants so that for all $(u,w),(u',w') \in \partial B_2 \times \partial B_3$ and all $z \in \Sigma$ we have

$$c_1 g_{(u,w)}(z) \le g_{(u',w')}(z) \le c_2 g_{(u,w)}(z).$$

Lemma 3.15. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$ we have for all $z \in \Sigma$*

$$\mathbb{P}(|G_n(z) - n \cdot \overline{g}(z)| \ge \varepsilon n \overline{g}(z)) \le Cn \exp(-c\sqrt{\varepsilon^2 n}) \quad \textit{and}$$
$$\mathbb{P}\Big(|\widetilde{G}_n(z) - n \cdot \overline{g}(z)| \ge \varepsilon n \overline{g}(z)\Big) \le C \exp(-c\sqrt{\varepsilon^2 n}).$$

Proof. We present a sketch of the proof for the random walk case.

Let N to be determined later. Then ζ_i is completely mixed by this time and so is $\widetilde{\zeta}$. We then have

$$\mathbb{P}(G_n(z) \ge (1+\varepsilon)n \cdot \overline{g}(z)) \le \mathbb{P}\left(\sum_{i=1}^{N} \xi_i g_{\zeta_i}(z) \ge n\varepsilon \overline{g}(z)/2\right) + N \max_{j \le N} \mathbb{P}\left(\sum_{i=1}^{n/N-1} g_{\zeta_{iN+j}}(z) \ge (1+\varepsilon/2) n \overline{g}(z)/N\right).$$

For every j, we can couple $(\zeta_{iN+j})_{i \le n/N-1}$ with i.i.d. points distributed according to e_{B_2,B_3}. The probability that the coupling fails can be bounded by ne^{-N}. We now want to apply Chernoff. For every $\lambda < c$ for some positive constant c coming from Harnack's inequality using the moment generating function of the exponential distribution we get

$$\mathbb{E}\Big[e^{\lambda \xi g_\zeta(z)}\Big] = \mathbb{E}\left[\frac{1}{1-\lambda g_\zeta(z)}\right] \le 1 + \lambda \overline{g}(z) + 2\lambda^2 \mathbb{E}\big[(g_\zeta(z))^2\big].$$

By Chernoff we get

$$\mathbb{P}\left(\sum_{i=1}^{n/N-1} \xi_{iN+j} \cdot g_{\zeta_{iN+j}}(z) \ge (1+\varepsilon) n \overline{g}(z)/N\right) \le \exp\left(\theta \overline{g}(z) + 2\theta^2 \mathbb{E}\big[(g_\zeta(z))^2\big]\right) \exp(-(1+\varepsilon)\theta n \overline{g}(z)/N).$$

Optimising over θ we get

$$\mathbb{P}\left(\sum_{i=1}^{n/N-1} \xi_{iN+j} \cdot g_{\zeta_{iN+j}}(z) \ge (1+\varepsilon) n \overline{g}(z)/N\right) \le \exp\left(-c \cdot \varepsilon^2 n/N\right) + e^{-N}$$

Taking N so that $\varepsilon^2 n = N^2$, we get the upper bound of order $\exp(-c\sqrt{\varepsilon^2 n})$ and concludes the proof.

The lower bound follows in the same way this time using for any $\lambda > 0$

$$\mathbb{E}\Big[e^{-\lambda \xi g_\zeta(z)}\Big] = \mathbb{E}\left[\frac{1}{1+\lambda g_\zeta(z)}\right] \le 1 - \lambda \overline{g}(z) + 2\lambda^2 \mathbb{E}\big[(g_\zeta(z))^2\big].$$

We leave the details to the reader. $\square$

3.3.4 Excursions of RW and RI

We define the capacity of A with respect to B and write $\mathrm{Cap}_B(A)$ as

$$\mathrm{Cap}_B(A) = \sum_{x\in A} \mathbb{P}_x\left(\widetilde{H}_A > H_B\right).$$

Lemma 3.16. *The expected length of each excursion across $B_3 \setminus B_2$ in $\mathbb{Z}_n^d$ in stationarity (for the clothesline) is given by*

$$\mathbb{E}_{e_{B_2,B_3}}[T_1] = \frac{n^d}{\mathrm{Cap}_{\partial B_3}(B_2)}.$$

Proof. We follow the proof of Cerny and Teixeira [24]. Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary two-sided random walk on $\mathbb{Z}_n^d$, i.e. X_0 is uniformly distributed and $(X_n)_{n\geq 0}$ and $(X_{-n})_{n\geq 0}$ are conditionally independent simple random walks on $\mathbb{Z}_n^d$ given X_0. We now define

$$\mathcal{R} = \{n \in \mathbb{Z} : X_n \in B_2 \text{ and } \exists\, m < n : X_m \in \partial B_3, \{X_{m+1}, \dots, X_n\} \subseteq B_3 \setminus B_2\},$$

i.e. $\mathcal{R}$ is the set of return times to ∂B_2 for the two sided random walk. By reversibility and stationarity we then get for $n \in \mathbb{N}$ and $x \in \partial B_2$

$$\begin{aligned}\mathbb{P}(n \in \mathcal{R}, X_n = x) &= \mathbb{P}(X_n = x, \exists\, m < n, X_m \in \partial B_3, \{X_{m+1}, \dots, X_n\} \subseteq B_3 \setminus B_2\}) \\ &= \frac{1}{n^d} \cdot \mathbb{P}_x\left(\widetilde{H}_{B_2} > H_{\partial B_3}\right).\end{aligned}$$

Taking the sum over all $x \in B_2$ we get

$$\mathbb{P}(n \in \mathcal{R}) = \frac{\mathrm{Cap}_{\partial B_3}(B_2)}{n^d}.$$

We now see that by taking the sum over all $n \in [0, t] \cap \mathbb{N}$ we get that the total number of excursions satisfies as $t \to \infty$

$$\frac{\mathbb{E}[\mathcal{N}_{\mathrm{RW}}(t)]}{t} \to \frac{\mathrm{Cap}_{\partial B_3}(B_2)}{n^d}.$$

Note that this limit is independent of the starting distribution of the walk and notice that the actual number of excursions up to time t can differ from the number above by at most 1 for every t, when considering only the one-sided walk.

Using now the ergodic theorem and the fact that the stationary distribution of the clothesline process is given by e_{B_2,B_3} concludes the proof. □

Lemma 3.17. *Let V be the total length of an excursion across $B_3 \setminus B_2$. There for all $k \geq 1$ we have*

$$\mathbb{E}\left[V^k\right] \leq k! \cdot \sup_{x\in\mathbb{Z}_n^d} \mathbb{E}_x[V].$$

Proof. Following [18] we write

$$\mathbb{E}\left[V^k\right] = \mathbb{E}\left[\int_0^\infty \dots \int_0^\infty \mathbf{1}(V > t_1) \cdots \mathbf{1}(V > t_k)\, dt_1 \dots dt_k\right] \tag{3.1}$$

$$\leq k! \cdot \int_{t_1\leq\ldots\leq t_k} \mathbb{P}(V > t_k, V > t_{k-1})\ dt_1 \dots dt_k. \tag{3.2}$$

Using the strong Markov property, we now bound

$$\int_{t_k \geq t_{k-1}} \mathbb{P}(V > t_k, V > t_{k-1})\, dt_k \leq \sup_{x \in \mathbb{Z}_n^d} \mathbb{E}_x[V] \cdot \mathbb{P}(V > t_{k-1})\,.$$

Iterating this and substituting in (3.1) establishes the desired bound. ☐

Lemma 3.18. *There exist positive constants c and C so that the following holds. For all $u > 0$ we have*

$$\mathbb{P}\Big(|\mathcal{N}_{\mathrm{RW}}(un^d) - u\mathrm{Cap}_{B_3}(B_2)| \geq \varepsilon u\mathrm{Cap}_{B_3}(B_2)\Big) \leq C \cdot \exp\left(-c\sqrt{\varepsilon^2 \cdot u \cdot \mathrm{Cap}_{B_3}(B_2)}\right)$$
$$\mathbb{P}\big(|\mathcal{N}_{\mathrm{RI}}(u) - u\mathrm{Cap}_{B_3}(B_2)| \geq \varepsilon u\mathrm{Cap}_{B_3}(B_2)\big) \leq C \cdot \exp\left(-c \cdot \varepsilon^2 \cdot u \cdot \mathrm{Cap}_{B_3}(B_2)\right)$$

4 Uniform spanning trees

4.1 Electrical networks

In this section we are going to give a short introduction to electrical network theory and how we can apply it to the study of random walks and uniform spanning trees.

Let $G = (V, E)$ be a finite connected graph with V the set of vertices and E the set of edges. We assign nonnegative weights $(w(e))_{e \in E}$ to the edge set E. We let a, b be two distinct vertices of V. We can view G as an electrical network by assigning resistances to the edges of the graph with $r(e) = w(e)^{-1}$. Talking from the electrical network point of view, we can imagine attaching a battery between a and b that applies a voltage difference which then induces a current that flows from a to b. To make it all mathematically rigorous, we define the voltage W to be a discrete harmonic function on $V \setminus \{a, b\}$, i.e. a function which satisfies for all $x \in V \setminus \{a, b\}$ that

$$W(x) = \frac{1}{w(x)} \cdot \sum_{y \sim x} w(x, y) W(y),$$

where $w(x) = \sum_{y \sim x} w(x, y)$. The voltage W induces a current that is defined by

$$I(x, y) = \frac{W(x) - W(y)}{r(x, y)}.$$

Then clearly I is an anti-symmetric function and satisfies for all $x \in V \setminus \{a, b\}$

$$\sum_{y \sim x} I(x, y) = 0.$$

This is called Kirchoff's node law. It also satisfies the cycle law, that is if $\overrightarrow{e}_1, \ldots, \overrightarrow{e}_k$ is an oriented cycle, then

$$\sum_{i=1}^{k} r(e_i) \cdot I(\overrightarrow{e}_i) = 0.$$

Let W_0 be a voltage with $W_0(a) = 1$ and $W_0(b) = 0$. By the uniqueness property of harmonic functions, any other harmonic function can be expressed as an affine function of W_0. More precisely, we can write

$$W(x) = (W(a) - W(b)) W_0(x) + W(b).$$

Let I_0 and I be the currents associated with W_0 and W respectively. We set

$$\|I_0\| = \sum_{x\sim a} I_0(a,x) \quad \text{and} \quad \|I\| = \sum_{x\sim a} I(a,x).$$

We then see that

$$\begin{aligned}\|I\| &= \sum_{x\sim a} I(a,x) = \sum_{x\sim a} \frac{W(a)-W(x)}{r(a,x)} = \sum_{x\sim a} \frac{1}{r(a,x)} \cdot (W(a)-W(b)-(W(a)-W(b))W_0(x)) \\ &= \sum_{x\sim a}(W(a)-W(b)) \cdot \frac{W_0(a)-W_0(x)}{r(a,x)} = (W(a)-W(b))\|I_0\|,\end{aligned}$$

which shows that the ratio

$$\frac{W(a)-W(b)}{\|I\|} = \frac{1}{\|I_0\|}$$

is independent of the voltage. We then define the effective resistance between a and b to be

$$R_{\text{eff}}(a,b) = \frac{W(a)-W(b)}{\|I\|}.$$

Lemma 4.1. *Let X be a simple random walk on G. Then*

$$\mathbb{P}_a\Big(H_b < \widetilde{H}_a\Big) = \frac{1}{\deg(a) R_{\text{eff}}(a,b)}.$$

Proof. The statement follows as a consequence of the uniqueness of harmonic functions on finite graphs with given boundary values. ☐

Definition 4.2. A flow $\theta : \overrightarrow{E} \to \mathbb{R}$ from a to b is an antisymmetric function defined on oriented edges satisfying for all $x \notin \{a,b\}$

$$\sum_{y\sim x} \theta(x,y) = 0 \quad \text{and} \quad \|\theta\| = \sum_{z\sim a} \theta(a,z) \geq 0.$$

We call θ a unit flow from a to b if $\|\theta\| = 1$.

Exercise 4.3. Let θ be a unit flow from a to b satisfying Kirchoff's node and cycle laws. Show that θ is the unit current flow from a to b.

Theorem 4.4 (Thomson's principle)**.** *The effective resistance satisfies*

$$R_{\text{eff}}(a,b) = \inf\left\{\sum_{e\in E} r(e)|\theta(e)|^2 : \ \theta \textit{ unit flow from } a \textit{ to } b\right\}.$$

This infimum is attained by the unit current flow from a to b.

Proof. We follow [7].

We write $\mathcal{E}(\theta) = \sum_{e\in E} r(e)|\theta(e)|^2$. Let i be the unit current flow from a to b associated to the potential φ.

We start by showing that

$$R_{\text{eff}}(a,z) = \mathcal{E}(i).$$

Using that i is a flow from a to z and Ohm's law we have

$$\mathcal{E}(i) = \frac{1}{2}\sum_{\substack{u,v\\u\sim v}} i(u,v)^2 r(u,v) = \frac{1}{2}\sum_{\substack{u,v\\u\sim v}} i(u,v)(\varphi(u)-\varphi(v)) = \varphi(a)-\varphi(z) = R_{\text{eff}}(a,z).$$

Let j be another flow from a to z of unit strength. The goal is to show that $\mathcal{E}(j) \geq \mathcal{E}(i)$.

We define $k = j - i$. Then this is a flow of 0 strength. So we now get

$$\begin{aligned}\mathcal{E}(j) &= \sum_e (j(e))^2 r(e) = \sum_e (i(e)+k(e))^2 r(e)\\ &= \sum_e (i(e))^2 r(e) + \sum_e (k(e))^2 r(e) + 2\sum_e k(e)i(e)r(e)\\ &= \mathcal{E}(i) + \mathcal{E}(k) + 2\sum_e k(e)i(e)r(e).\end{aligned}$$

We now show that

$$\sum_e k(e)i(e)r(e) = 0.$$

Since i is the unit current flow associated with φ, for $e=(x,y)$ it satisfies

$$i(x,y) = \frac{\varphi(x)-\varphi(y)}{r(x,y)}.$$

Substituting this above we obtain

$$\sum_e k(e)i(e)r(e) = \frac{1}{2}\cdot\sum_x\sum_{y\sim x}(\varphi(x)-\varphi(y))k(x,y) = \frac{1}{2}\cdot\sum_x\sum_{y\sim x}\varphi(x)k(x,y) + \frac{1}{2}\cdot\sum_x\sum_{y\sim x}\varphi(x)k(x,y),$$

where for the last equality we used the antisymmetric property of k. Since k is a flow of 0 strength, we get that both these sums are equal to 0. Therefore this proves that

$$\mathcal{E}(j) \geq \mathcal{E}(i)$$

with equality if and only if $\mathcal{E}(k) = 0$ which is equivalent to $k = 0$. □

Corollary 4.5 (Rayleigh's monotonicity principle)**.** *The effective resistance is a monotone function of the component resistances.*

4.2 Uniform spanning trees on finite graphs

Let $G = (V, E)$ be a finite connected graph with V being the set of vertices and E the set of edges.

Definition 4.6. A **spanning tree** of G is a connected subgraph of G which is a tree (i.e. it contains no cycles) and contains all vertices of G. Since G is finite, the total number of spanning trees of G is finite. Let this collection be $\mathcal{T}$. An element $T \in \mathcal{T}$ picked uniformly at random is called a **uniform spanning tree (UST)**.

Let $\mathcal{N}(s,a,b,t)$ be the set of spanning trees of G with the property that the unique path from s to t passes along the edge (a,b) in the direction from a to b. We write $N(s,a,b,t) = |\mathcal{N}(s,a,b,t)|$.

Let N be the total number of spanning trees of G. We then have the following theorem:

Theorem 4.7. *The function*

$$i(a,b) = \frac{N(s,a,b,t) - N(s,b,a,t)}{N}$$

for all $(a,b) \in E$ defines a unit flow from s to t satisfying Kirchoff's laws (node and cycle laws).

Proof. It is obvious from definition that i is an antisymmetric function. We next check that it satisfies Kirchoff's node law, i.e. for all $a \notin \{s,t\}$ we have

$$\sum_{x \sim a} i(a,x) = 0.$$

We now count the contribution of each spanning tree T to the sum above. We now consider the unique path from s to t in this spanning tree. If a is a vertex on this path, then there are two edges on the path with endpoint a that contribute to the sum. The edge going into a and the one going out of a. The first one will contribute $-1/N$ and the second one $1/N$. Now if a is not on the path, then there is no contribution to the sum from T. Hence the overall contribution of T is $-1/N + 1/N = 0$ and this proves Kirchoff's node law.

We now check that it satisfies the cycle law. Let $v_1, \ldots, v_n, v_{n+1} = v_1$ constitute a cycle C. We will show that

$$\sum_{i=1}^{n} i(v_i, v_{i+1}) = 0. \tag{4.1}$$

To do this we will work with *bushes* instead of trees. We define an s/t bush to be a forest consisting of exactly two trees T_s and T_t such that $s \in T_s$ and $t \in T_t$. Let $e = (a,b)$ be an edge. We define $\mathcal{B}(s,a,b,t)$ as the set of s/t bushes with $a \in T_s$ and $b \in T_t$.

We now claim that $|\mathcal{B}(s,a,b,t)| = N(s,a,b,t)$. Indeed, for every bush in $\mathcal{B}(s,a,b,t)$ by adding the edge e we obtain a spanning tree of $\mathcal{N}(s,a,b,t)$. Also for every spanning tree $T \in \mathcal{N}(s,a,b,t)$ by removing the edge e we obtain a bush in $\mathcal{B}(s,a,b,t)$.

So instead of counting the contribution of each spanning tree to the sum in (4.1) we look at bushes. Let B be an s/t bush. Then B makes a contribution to $i(a,b)$ of $1/N$ if $B \in \mathcal{B}(s,a,b,t)$, $-1/N$ if $B \in \mathcal{B}(s,b,a,t)$ and 0 otherwise.

So in total an s/t bush B contributes $(F_+ - F_-)/N$, where F_+ is the number of pairs (v_j, v_{j+1}) so that $B \in \mathcal{B}(s, v_j, v_{j+1}, t)$ and similarly for F_-.

But since C is a cycle, if there is a pair (v_j, v_{j+1}) in F_+, then there must be a pair (v_i, v_{i+1}) in F_-. (If not, this would violate the no cycle property of the tree.) Therefore we get $F_+ = F_-$ and hence the total contribution of B is 0.

Finally we need to check that i is a unit flow, i.e.

$$\sum_{x \sim s} i(s,x) = 1.$$

First we note that $N(s,x,s,t) = 0$ for all $x \sim s$. Every spanning tree must contain a path from s to t, and hence this gives that

$$\sum_{x \sim s} N(s,s,x,t) = N$$

and concludes the proof. $\square$

Corollary 4.8. *Let $e = (s,t)$ be an edge of G and let T be a UST. Then*

$$R_{\text{eff}}(e) = \mathbb{P}(e \in T)\,.$$

Proof. Combining Theorem 4.7 and Exercise 4.3 we get that i from Theorem 4.7 is a unit current flow from s to t and

$$i(s,t) = \frac{N(s,s,t,t)}{N},$$

where $N(s,s,t,t)$ is the number of spanning trees that use the edge (s,t). Hence

$$\frac{N(s,s,t,t)}{N} = \mathbb{P}(e \in T)\,.$$

Since the network has unit conductances, we get that

$$i(s,t) = \varphi(s) - \varphi(t),$$

where φ is the voltage associated to the unit current i. Therefore the effective resistance between s and t is given by

$$R_{\text{eff}}(s,t) = i(s,t) = \mathbb{P}(e \in T)$$

and this completes the proof. $\square$

Theorem 4.9. *Let $G = (V,E)$ be a finite graph. Let $f, g \in E$ with $f \neq g$. Let T be a UST. Then*

$$\mathbb{P}(f \in T \mid g \in T) \leq \mathbb{P}(f \in T)\,.$$

Proof. We consider G as a network with every edge having conductance 1. From Corollary 4.8 we get that

$$R_{\text{eff}}(f;G) = \mathbb{P}(f \in T)\,,$$

where we write $R_{\text{eff}}(f;G)$ to specify that the effective resistance is considered in the graph G. Let e and g be distinct edges of G. We write $G.g$ for the graph obtained by gluing both endpoints of g to a single vertex. In this way we obtain a one to one correspondence between spanning trees of G containing g and spanning trees of $G.g$. Therefore, $\mathbb{P}(e \in T \mid g \in T)$ is the proportion of spanning trees of $G.g$ containing e. So from the above

$$\mathbb{P}(f \in T \mid g \in T) = R_{\text{eff}}(f; G.g).$$

Gluing the two endpoints of g decreases the effective resistance by Rayleigh's principle, and hence

$$R_{\text{eff}}(f; G.g) \leq R_{\text{eff}}(f; G),$$

which is exactly the statement of the theorem. $\square$

Definition 4.10. Let G be a finite connected graph. We write $\mathcal{F}$ for the set of forests of G (subsets of G that do not contain cycles). Let F be a forest picked uniformly at random among all forests in $\mathcal{F}$. We refer to it as USF.

Conjecture 4.11. *For $f, g \in E$ with $f \neq g$ the USF satisfies*

$$\mathbb{P}(f \in F \mid g \in F) \leq \mathbb{P}(f \in F)\,.$$

There is a computer aided proof (Grimmett and Winkler) which shows that for graphs on 8 or fewer vertices this conjecture is true.

4.3 Wilson's algorithm

In order to understand the geometry of UST's, it turns out it is useful to study sampling algorithms.

In this section we describe a beautiful algorithm due to David Wilson.

We start by first describing the loop erasure of a finite path. Let $G = (V, E)$ be a finite graph and let $\gamma = (\gamma_0, \ldots, \gamma_n)$ be a finite path of vertices in G. The loop erasure of γ is the path obtained when we remove loops from γ in the chronological order in which they were created. More precisely, we define

$$i_0 = \sup\{j : \gamma_j = \gamma_0\}$$

and for $k \geq 1$ we define inductively

$$i_k = \sup\{j : \gamma_j = \gamma_{i_{k-1}+1}\}.$$

Let $m = \inf\{j : i_j = n\}$. Then the path The path $(\gamma_{i_k})_{k \leq m}$ obtained in this way is called the loop-erasure of γ.

We can now describe Wilson's algorithm. Fix an ordering of the vertex set $V = \{v_0, v_1, \ldots, v_n\}$ and set $r = v_0$ to be the root and set $T_0 = \{r\}$. Next start a simple random walk from v_1 and run it until it hits r. Then erase the loops and add the loop erased path to T_0 to obtain T_1. Inductively, once T_i has been defined, start an independent simple random walk from the next vertex in the ordering and run it until it hits T_i. Then erase the loops and append the new path to T_i in order to obtain T_{i+1}. We continue like this until we exhaust the vertices of V.

Theorem 4.12 (Wilson's algorithm)**.** *Wilson's algorithm produces a uniform spanning tree.*

In order to prove that Wilson's algorithm works, we will first describe an equivalent way of running the algorithm.

Under every vertex of the graph except for the root we place an infinite collection of cards with i.i.d. instructions on them. Each card points to a neighbour of the vertex chosen uniformly at random. Different cards contain independent instructions. At time 1 we reveal all the top cards under every vertex. These give rise to a directed graph. If there is no directed cycle, then we have obtained a spanning tree. If there is a directed cycle, then we remove the top cards under every vertex on the cycle and we reveal the second cards. Every time a directed cycle appears we remove the corresponding cards and reveal the next ones. Discarding the top cards under a cycle is called cycle popping. We continue in this way until there are no cycles. We will show that this procedure will stop with probability 1 and the resulting tree will have the distribution of a uniform spanning tree.

Lemma 4.13. *Given any instructions under the vertices of the graph, either any order of popping cycles will pop an infinite number of cycles or if not, it will always result in the same spanning tree.*

Proof. First of all we introduce colours to the directed edges that come from the instructions. The edges that appear from the top cards get colour 1, while the edges that appear from the i-th cards get colour i.

Suppose that C is a cycle that can be popped in the order $C_1, C_2, \ldots, C_n = C$ and suppose that a different coloured cycle $C' \neq C_1$ is popped first. We want to show that either $C' = C$ or else C will still be popped after C' is popped. This will suffice as it will prove that either an infinite number of cycles will be popped or every cycle will be popped. If C' has no vertices in common with any of $C_1, \ldots, C_n$, then C can still be popped. If on the other hand C' has vertices in common with $C_1, \ldots, C_n$, then let k be the first index so that $C' \cap C_k \neq \emptyset$ and let $x \in C' \cap C_k$. Since $x \notin C_1, \ldots, C_{k-1}$, it follows that the edge that comes out of x is of colour 1, as it has not been discarded yet. Also all the edges of C' are of colour 1, as it is the first cycle that we are popping. The edge coming out of x will thus lead to the same point in both C' and C_k. Continuing in this way, we get that all the colours of the new edges will also be 1, and hence we arrive at the conclusion that $C' = C_k$. This shows that if $C' \neq C$, then we can still pop C in the order $C', C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_n = C$. $\square$

Proof of Wilson's algorithm. First of all we observe that if we erase loops in the order in which they are created, this is one way of popping cycles. Since Wilson's algorithm stops with probability 1, it follows that any other method of popping cycles will also stop with probability 1 and will produce a spanning tree. We now explain why the distribution of the tree is the uniform one.

We are going to consider pairs of sets of cycles and spanning trees lying underneath them. Let $X = \{(O, T) : O \text{ set of cycles lying over } T \text{ (a spanning tree)}\}$. Then for any finite set of cycles O, any spanning tree T could be lying underneath, since the instructions under the vertices in O could be anything. Therefore, we see that the space X can be written as $X = X_1 \times X_2$, where X_1 is the set of coloured cycles and X_2 is the set of all spanning trees of G. Now the probability of seeing a spanning tree T and a set of cycles O on top of it is given by

$$\prod_{e \in \cup(O \cup T)} \frac{1}{\deg(e^-)} = \prod_{e \in \cup O} \frac{1}{\deg(e^-)} \cdot \prod_{e \in T} \frac{1}{\deg(e^-)}.$$

This shows that the marginal distribution of the tree component is the uniform distribution on the space of spanning trees, as

$$\prod_{e \in T} \frac{1}{\deg(e^-)} = \prod_{x \neq r} \frac{1}{\deg(x)}.$$

This completes the proof. $\square$

Theorem 4.14. *Let G be an finite connected graph and let T be a UST. Let x, y be two distinct vertices of G. Then the unique path joining x and y in T has the distribution of a loop-erased random walk started from x run until it hits y. Moreover, a loop-erased random walk from x to y has the same distribution as a loop-erased random walk from y to x in the graph G.*

Proof. This follows immediately from Wilson's algorithm, as we can take x to be the root and start a random walk first from y and run it until it hits x. $\square$

4.4 Uniform spanning forests

In this section we define the uniform spanning forest measure in infinite graphs.

First of all we see that if a graph is recurrent, then we can run Wilson's algorithm as in the finite case and this will produce a spanning tree. It turns out that if we restrict this tree to large finite balls, then the distribution will be close to the UST distribution in the finite graph. What happens if we have an infinite transient graph though? We will show that one can always define a USF measure on infinite graphs irrespectively of whether they are recurrent or transient. Then we will present sampling algorithms for the transient case as well.

Theorem 4.15. *There exists a measure μ supported on spanning forests of $\mathbb{Z}^d$ for all $d \geq 1$. We call μ the (wired) USF measure on $\mathbb{Z}^d$.*

Sketch of proof. Let $(B_n)_n$ be an exhaustion of $\mathbb{Z}^d$ by finite graphs, i.e. $B_n \subseteq B_{n+1}$ for all $n \geq 0$ and $\cup_n B_n = \mathbb{Z}^d$. We obtain a sequence of graphs G_n by gluing B_n^c into a single point ∂_n and keeping all the existing connections in B_n and between ∂_n and B_n. Let μ_n be the UST measure on G_n. Let $e_1, \ldots, e_k$ be a finite collection of edges such that for all n sufficiently large $e_i \in B_n$ for all $i \leq k$. For $T \sim \mu_n$ we have

$$\mu_n(e_1, \ldots, e_k \in T) = \mathbb{P}(e_1, \ldots, e_k \in T) = \mathbb{P}(e_k \in T \mid e_1, \ldots, e_{k-1} \in T)\, \mathbb{P}(e_1, \ldots, e_{k-1} \in T)\,.$$

From the proof of Theorem 4.9 we get

$$\mu_n(e_1, \ldots, e_k \in T) = \mathbb{P}(e_k \in T \mid e_1, \ldots, e_{k-1} \in T) = R_{\text{eff}}(e_k; G_n/\{e_1, \ldots, e_{k-1}\}),$$

where we write $G_n/\{e_1, \ldots, e_{k-1}\}$ for the graph G_n with all the endpoints of the edges $e_1, \ldots, e_{k-1}$ glued to single points. By Rayleigh's monotonicity principle, we now see that

$$R_{\text{eff}}(e_k; G_n/\{e_1, \ldots, e_{k-1}\}) \leq R_{\text{eff}}(e_k; G_{n+1}/\{e_1, \ldots, e_{k-1}\}).$$

Therefore, we see that the sequence $\mu_n(e_1, \ldots, e_k \in T)$ is an increasing sequence, and hence has a limit $\mu(e_1, \ldots, e_k \in \mathcal{F})$, where $\mathcal{F}$ denotes a random forest. From this it also follows that the limiting measure μ is supported on acyclic graphs. Using this and the inclusion-exclusion formula, we can define μ on elementary cylinder sets, i.e. sets of the form

$$\{F \subseteq \mathbb{Z}^d : F \text{ forest },B_1 \subseteq F \text{ and } B_2 \cap F = \emptyset\},$$

where B_1 and B_2 are finite sets of edges. Using next Kolmogorov's extension theorem we get that there is a uniquely defined measure μ supported on acyclic subgraphs of $\mathbb{Z}^d$ that is the weak-limit of the sequence (μ_n). We leave it to the reader to check using Rayleigh's principle that the limit measure μ is independent of the choice of exhaustion. $\square$

Wilson's method rooted at infinity [4] Let G be a transient graph. We set $\mathcal{F}_0 = \emptyset$ and assign an ordering to the vertices of G. We then start a simple random walk from the first vertex of the ordering and run it forever and erase loops (which is well defined as it is almost surely a transient trajectory). We then define $\mathcal{F}_1$ to be this loop-erased path. Inductively, once we have defined $\mathcal{F}_i$ we define $\mathcal{F}_{i+1}$ by running an independent simple random walk from the next vertex in the ordering and wait until it hits $\mathcal{F}_i$. If it never hits $\mathcal{F}_i$, then we run it indefinitely. In both cases we erase loops from the obtained path and we enlarge $\mathcal{F}_i$ by appending this path to it to get $\mathcal{F}_{i+1}$. We continue until we exhaust the vertex set of G.

Theorem 4.16 ([4])**.** *Wilson's method rooted at infinity yields a USF in $\mathbb{Z}^d$.*

Sketch of proof. Let B_n be an exhaustion of $\mathbb{Z}^d$ by finite graphs and let G_n be the graph obtained by gluing B_n^c to a single point and keeping all connections. We run Wilson's algorithm on the graph G_n by taking as the root the vertex ∂_n. Every time the walks used in Wilson's algorithm hit ∂_n we stop them. This way we can couple all the walks on the different graphs. The loop erasure of each such walk will converge to the infinite loop erasure of a random walk. Using that Wilson's algorithm yields a UST in the finite setting together with the above shows that Wilson's method rooted at infinity will also produce a USF. $\square$

4.4.1 Connectivity of USF

Theorem 4.17 (Pemantle 1991). *The uniform spanning forest of an infinite graph is a tree if and only if a simple random walk and an independent loop erased random walk started from any two distinct vertices intersect with probability* 1. *Moreover, the probability that* x *and* y *are in the same tree of the USF is equal to the probability that a random walk started from* x *intersects an independent loop-erased random walk started from* y.

Proof. This is immediate using Wilson's method rooted at infinity. $\square$

Theorem 4.18 (Lyons, Peres and Schramm [15]). *Fix* $k \geq 0$ *and let* $(x_j)_{j=-k,\ldots,-1}$ *be a path in* $\mathbb{Z}^d$. *Let* X *and* Y *be two independent simple random walks in* $\mathbb{Z}^d$ *started from* x_0 *and* y_0 *respectively. Let* $I = \sum_{i,j=0}^{\infty} \mathbf{1}(X_i = Y_j)$. *If* $\mathbb{E}_{(x,y)}[I] = \infty$ *for all* x, y, *then setting* $X_j = x_j$ *for* $j = -k, \ldots, -1$ *we have*

$$\mathbb{P}(|\mathrm{LE}(X_j)_{j \geq -k} \cap \{Y_n : n \geq 0\}| = \infty) \geq \frac{1}{16}.$$

Let X and Y be two independent simple random walks. For every N we write

$$I_N = \sum_{i,j=0}^{N} \mathbf{1}(X_i = Y_j)$$

for the total number of intersections up to time N.

Lemma 4.19. *For all* $x, y \in \mathbb{Z}^d$ *we have*

$$\liminf_{N \to \infty} \frac{(\mathbb{E}_{x,y}[I_N])^2}{\mathbb{E}_{(x,y)}\left[I_N^2\right]} \geq \frac{1}{4}.$$

Proof. We first prove this lemma for $x = y = 0$. Writing g_N for the Green kernel up to time N we have

$$\begin{aligned}
\mathbb{E}_{(0,0)}\left[I_N^2\right] &= \sum_{i,j,k,l=0}^{N} \sum_{x,y} \mathbb{P}_{(0,0)}(X_i = Y_k = x, X_j = Y_\ell = y) \\
&\leq \sum_{x,y} \left(\sum_{i \leq j \leq N} \mathbb{P}_0(X_i = x)\, \mathbb{P}_x(X_{j-i} = y) + \sum_{N \geq i > j} \mathbb{P}_0(X_j = y)\, \mathbb{P}_y(X_{i-j} = x) \right) \\
&\qquad \times \left(\sum_{k \leq \ell \leq N} \mathbb{P}_0(Y_k = x)\, \mathbb{P}_x(Y_{\ell-k} = y) + \sum_{N \geq k > \ell} \mathbb{P}_0(Y_\ell = y)\, \mathbb{P}_y(Y_{k-\ell} = x) \right) \\
&\qquad \leq \sum_{x,y} (g_N(x) g_N(x-y) + g_N(y) g_N(x-y))^2 \\
&\qquad \leq 2 \sum_{x,y} (g_N(x)^2 g_N(x-y)^2 + g_N(y)^2 g_N(x-y)^2) = 4(\mathbb{E}_{(0,0)}[I_N])^2,
\end{aligned}$$

where for the last inequality we used that $(a+b)^2 \leq 2(a^2 + b^2)$.

For $x \neq y$, the same argument as above shows that

$$\mathbb{E}_{(x,y)}\left[I_N^2\right] \leq 4(\mathbb{E}_{(0,0)}[I_N])^2.$$

We only need to show that $\liminf \mathbb{E}_{(x,y)}[I_N] / \mathbb{E}_{(0,0)}[I_N] \geq 1/4$. For this we refer the reader to [14, Corollary 10.32]. $\square$

Proof of Theorem 4.18. For every m we set $(L_j^m)_{j=0}^{J(m)} = \mathrm{LE}(X_0,\ldots,X_m)$, where $J(m)$ is the length of the loop erasure. On the event $\{X_m = Y_n\}$, we define

$$i(m,n) = \min\{i \geq 0 : L_i^m \in \{Y_n, Y_{n+1},\ldots\}\} \text{ and}$$
$$j(m,n) = \min\{j \geq 0 : L_j^m \in \{X_m, X_{m+1},\ldots\}\}.$$

Then $J(m)$ belongs to both sets of which we are taking the minimum. When $X_m \neq Y_n$, we set $i(m,n) = j(m,n) = 0$. We now notice that on the event $\{X_m = Y_n\}$ we have

$$(X_m, X_{m+1},\ldots) \stackrel{d}{=} (Y_n, Y_{n+1},\ldots).$$

This then gives that

$$\mathbb{P}(i(m,n) \leq j(m,n) \mid X_m = Y_n) \geq \frac{1}{2}. \tag{4.2}$$

Next we define

$$\widetilde{I}_N = \sum_{m=0}^{N}\sum_{n=0}^{N} \mathbf{1}(X_m = Y_n)\cdot \mathbf{1}(i(m,n) \leq j(m,n)) \leq I_N.$$

Therefore, we get $\mathbb{E}\Big[\widetilde{I}_N^2\Big] \leq \mathbb{E}\big[I_N^2\big]$. For the first moment of $\widetilde{I}_N$ we have using (4.2)

$$\mathbb{E}\Big[\widetilde{I}_N\Big] = \sum_{m=0}^{N}\sum_{n=0}^{N} \mathbb{P}(i(m,n) \leq j(m,n) \mid X_m = Y_n)\cdot \mathbb{P}(X_m = Y_n) \geq \frac{1}{2}\mathbb{E}[I_N]\,.$$

By the Payley-Zygmund inequality we then deduce for all $\varepsilon > 0$

$$\mathbb{P}\Big(\widetilde{I}_N \geq \varepsilon\mathbb{E}\Big[\widetilde{I}_N\Big]\Big) \geq (1-\varepsilon)^2\cdot\frac{\Big(\mathbb{E}\Big[\widetilde{I}_N\Big]\Big)^2}{\mathbb{E}\Big[\widetilde{I}_N^2\Big]} \geq (1-\varepsilon)^2\cdot\frac{1}{4}\cdot\frac{(\mathbb{E}[I_N])^2}{\mathbb{E}\big[I_N^2\big]} \geq \frac{1}{16}\cdot(1-\varepsilon)^2 - \varepsilon.$$

Since $\mathbb{E}[I_N] \to \infty$, it also follows that $\mathbb{E}\Big[\widetilde{I}_N\Big] \to \infty$ as $N \to \infty$ as well. Therefore, we deduce

$$\mathbb{P}\Big(\widetilde{I}_N \to \infty\Big) \geq \frac{1}{16}.$$

We make the observation that if $X_m = Y_n$ and $i(m,n) \leq j(m,n)$, then $L_{i(m,n)}^m \in \mathrm{LE}(X) \cap Y$. This together with the above imply

$$\mathbb{P}(|\mathrm{LE}(X) \cap Y| = \infty) \geq \frac{1}{16}$$

and concludes the proof. $\square$

Theorem 4.20. *The USF in $\mathbb{Z}^d$ is a single tree almost surely if and only if $d \leq 4$.*

Proof. We start with $d \geq 5$. Let X and Y be two independent simple random walks started from x and y respectively. Then the expected number of intersections is upper bounded by

$$\mathbb{E}_{(x,y)}[I] \lesssim \sum_{t=\|x-y\|}^{\infty} \frac{1}{t^{d/2-1}}.$$

We see that for every $\varepsilon > 0$, taking $\|x - y\|$ sufficiently large and $d \geq 5$ we get that $\mathbb{E}_{(x,y)}[I] \leq \varepsilon$. Thus,

$$\mathbb{P}(\text{USF is connected}) \leq \mathbb{P}_{(x,y)}(I > 0) \leq \mathbb{E}_{(x,y)}[I] \leq \varepsilon.$$

Since this is true for any $\varepsilon > 0$, it follows that for $d \geq 5$ we have

$$\mathbb{P}(\text{USF is connected}) = 0.$$

We now focus on $d \leq 4$. In this case we know that for all x, y we have

$$\mathbb{E}_{(x,y)}[I] = \infty.$$

Let $\Lambda = \{|\text{LE}(X) \cap Y| = \infty\}$. To show that the USF is connected almost surely, it suffices to prove that $\mathbb{P}_{(x,y)}(\Lambda) = 1$ for all x, y. By the martingale convergence theorem we get

$$\mathbb{P}_{(x,y)}(\Lambda \mid X_1, \ldots, X_n, Y_1, \ldots, Y_n) \to \mathbf{1}(\Lambda) \quad \text{as } n \to \infty \quad \text{a.s.}$$

By the Markov property of Y we also get

$$\mathbb{P}_{(x,y)}(\Lambda \mid X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \mathbb{P}_{(x,Y_n)}(\Lambda \mid X_1, \ldots, X_n)\,.$$

But applying Theorem 4.18 we get that this last probability is lower bounded by 1/16. Therefore, combining this with the convergence above, we get

$$\mathbf{1}(\Lambda) \geq \frac{1}{16},$$

and hence this implies that $\mathbb{P}_{(x,y)}(\Lambda) = 1$ and concludes the proof. ☐

4.5 Aldous Broder algorithm

Definition 4.21. A **rooted tree** is a tree together with a distinguished vertex that we call the root.

An **oriented rooted tree** is a rooted tree where every edge has an orientation with the property that for every vertex except for the root there is exactly one edge emanating from it pointing away from it.

Remark 4.22. We note that in an oriented rooted tree every edge is directed towards the root. Also for every rooted tree, there is a unique way of assigning orientations to the edges so that every edge points towards the root. In particular, for every edge $e = (x, y)$, we orient it from x to y if $d(r, x) > d(r, y)$, where r is the root of the tree and d is the graph distance on the tree. We orient it from y to x if $d(r, x) < d(r, y)$.

Aldous-Broder (with ideas from Diaconis) algorithm. Let $G = (V, E)$ be a finite connected graph and let X be a simple random walk on G started from some vertex x_0. We run the walk X up until the cover time, which is the first time that the walk has visited every vertex of the graph at least once, i.e. $\tau_{\text{cov}} = \max_x H_x$, where we recall that we write $H_x = \min\{t \geq 0 : X_t = x\}$ is the first hitting time of x. To construct an oriented spanning tree, for every vertex $x \neq x_0$, we include in the collection the edge that was crossed the first time that the walk visited x but with the reversed orientation. More precisely, for a directed edge e we write $e^{\leftarrow}$ for the edge with the opposite orientation. Then the Aldous Broder procedure applied to X is given by

$$\mathrm{AB}(X) = \{(X_{H_v - 1}, X_{H_v})^{\leftarrow} : v \neq x_0\}.$$

In this way, all the edges that we keep are oriented towards the root x_0 of the tree.

It follows immediately by the construction that $\mathrm{AB}(X)$ is an oriented spanning tree. Indeed, every vertex has exactly one directed edge emanating from it which points away from it. Moreover, the definition in terms of the first hitting time of every vertex forces it to be acyclic.

To prove that this algorithm produces a UST, we are going to show that the resulting tree has the same distribution as the invariant distribution of a certain Markov chain on the space of rooted oriented trees.

Markov chain on rooted oriented trees. Let T_0 be a rooted oriented tree rooted at X_0. To describe one step of the Markov chain, we choose a neighbour X_1 of X_0 uniformly at random. We then add the edge (X_0, X_1) with direction from X_0 to X_1. This now created a cycle, and hence to break it we delete the unique edge coming out of X_1 pointing away from it. This new tree T_1 is an oriented tree rooted at X_1.

Let Q be the transition matrix of this Markov chain. Then for two trees S, T such that $Q(S,T) > 0$ we have

$$Q(S,T) = \frac{1}{\deg(\rho(S))},$$

where $\rho(S)$ stands for the root of the tree S.

Exercise 4.23. Show that the Markov chain defined by Q is irreducible. Check that if $\pi(v) = \deg(v)/(2|E|)$, then $1/|\mathcal{T}| \cdot \pi$ is an invariant distribution for Q (considered as as Markov chain on the space $\mathcal{T} \times V$ (trees with their roots)).

Theorem 4.24 (Aldous-Broder (and Diaconis))**.** *For every x_0, if X is a simple random walk on G started from x_0, then* $\mathrm{AB}(X)$ *has the distribution of an oriented rooted UST of G.*

Proof. Let $(X_j : -\infty < j < \infty)$ be a stationary two-sided random walk on G, i.e. $X_0 \sim \pi$ and $(X_n)_{n \geq 0}$ and $(X_{-n})_{n \geq 0}$ are independent conditionally on X_0. Define for all $j \geq 0$

$$S_j = \mathrm{AB}(X_j, X_{j+1}, \ldots).$$

Then $(S_j)_j$ is a stationary sequence of rooted oriented trees, because $(X_j)_j$ is stationary. We next show that

$$S_0 \sim \frac{1}{|\mathcal{T}|} \cdot \pi,$$

where $|\mathcal{T}|$ is the total number of spanning trees and π is the degree biased distribution.

To prove this, we consider the reversed chain, i.e. the Markov chain with transition matrix

$$P(T, T') = \mathbb{P}\left(S_{-1} = T' \;\middle|\; S_0 = T\right),$$

for T, T' rooted oriented trees. Now for any rooted oriented tree T, there are $\deg(\rho(T))$ possible trees T' which differ in exactly two edges such that $P(T,T') > 0$. Since X_{-1} is a uniform neighbour of X_0, it follows that for any such compatible tree T' we have

$$P(T, T') = \frac{1}{\deg(\rho(T))}.$$

This now implies that the matrix P has the same invariant distribution as Q, which is $1/|\mathcal{T}|\cdot\pi$. Since this Markov chain is also irreducible, it follows that $1/|\mathcal{T}|\cdot\pi$ is the unique invariant distribution.

To finish the proof we note that by construction the sequence of trees (S_j) is a stationary sequence, and hence, S_0 must be distributed according to the invariant distribution. If we now condition on $X_0 = x_0$, then we get

$$\mathbb{P}(S_0 = T \mid X_0 = x_0) = \frac{1}{|\mathcal{T}|}\cdot\pi(x_0)\cdot\frac{1}{\pi(x_0)} = \frac{1}{|\mathcal{T}|}$$

and this concludes the proof. □

4.5.1 Ends in trees

Definition 4.25. An infinite path in a tree that never backtracks is called a **ray**. Two rays are equivalent if they have infinitely many vertices in common. An equivalence class of rays is called an **end**.

We already know that the USF is almost surely connected in $\mathbb{Z}^d$ when $d \leq 4$ and consists of infinitely many trees when $d \geq 5$. We would like to understand the geometry of each tree in the USF in all dimensions. In particular, how many ends does each tree have?

Theorem 4.26 (Pemantle ($d \leq 4$) and [4] ($d \geq 5$))**.** *All trees in the USF are almost surely one-ended for all $d \geq 2$.*

Remark 4.27. In the case of $d = 1$, the USF is the whole line, which is clearly two-ended.

Remark 4.28. A tree is one-ended if and only if it does not contain a bi-infinite path.

Remark 4.29. Theorem 4.26 for $d = 2$ is proved in [14] using a duality argument, which works more generally for plane recurrent graphs G whose dual graph is also recurrent and locally finite.

It turns out to be easier to talk about one-endedness when we assign orientations to the USF. We have already described Wilson's method rooted at infinity. We now also add orientations to the edges by assigning the direction that agrees with the direction in which the LERW crossed that edge. In this way every vertex has exactly one oriented edge emanating from it in the forest. Therefore, every vertex u will have a unique infinite oriented path emanating from it. We call this path the **future** of u. The set of vertices with an oriented path pointing towards u is called the past of u. The important observation is the following

$$v \in \text{future of } u \Leftrightarrow u \in \text{past of } v.$$

With this observation the following becomes immediate

Lemma 4.30. *A tree in the USF is one-ended if and only if every vertex has finite past.*

4.5.2 Aldous-Broder Interlacements

We have already seen how to extend Wilson's algorithm to infinite graphs. How can we extend the Aldous-Broder algorithm? In the case of an infinite recurrent graph, it is clear how to do it. Namely, we run a random walk forever and for every vertex we include in the spanning tree the first entry edge to that vertex. The recurrence of the graph implies that this algorithm will visit all vertices of the graph, and hence it will produce a spanning tree. How can we extend the algorithm though to an infinite transient graph?

Hutchcroft's idea was to replace the single walk by random interlacements and use the same algorithm. We explain this generalisation.

Let $\mathcal{I}$ be the random interlacements process, i.e. it is a Poisson process on the space $\mathcal{W}^* \times \mathbb{R}$, where $\mathcal{W}^*$ is the space of bi-infinite trajectories modulo time-shift. For every $t \in \mathbb{R}$ and $v \in \mathbb{Z}^d$ we define

$$\tau_t(v) = \inf\left\{s \geq t : \exists\, W \in \mathcal{W}^*_{\{v\}} \text{ s.t. } (W,s) \in \mathcal{I}\right\},$$

where we recall that $\mathcal{W}^*_{\{v\}}$ stands for the set of trajectories in $\mathcal{W}^*$ that hit v. It is clear by the construction of random interlacements that there exists a unique trajectory hitting v at time $\tau_t(v)$ and we denote it $W_{\tau_t(v)}$. Let $e_t(v)$ be the entry edge to v by $W_{\tau_t(v)}$ the first time that v was hit. Finally for every t we set

$$\mathrm{AB}_t(\mathcal{I}) = \left\{e_t(v)^{\leftarrow} : v \in \mathbb{Z}^d\right\}.$$

Theorem 4.31 (Hutchcroft [8]). *For all $t > 0$, the forest $\mathrm{AB}_t(\mathcal{I})$ has the law of an oriented USF of $\mathbb{Z}^d$.*

Proof. Let (B_n) be an exhaustion of $\mathbb{Z}^d$ by finite sets. Consider the graphs (G_n) obtained by gluing B_n^c to a single point ∂_n. Now for every n we let P_n be a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure. For every $t \in P_n$ we let W_t be a random walk excursion in G_n that starts and ends at ∂_n. We now claim that the process

$$\{(t, W_t) : t \in P_n\}$$

is a Poisson process that converges weakly to the random interlacements process on $\mathbb{Z}^d$. Indeed, if we define a measure for every finite set $K \subseteq B_n$

$$\begin{aligned} &Q_K^n(\{w \in \mathcal{W}[-k,m] : w\,|_{[-k,0]} \in A, w(0) = u, w\,|_{[0,m]} \in B\}) \\ &= \mathbb{P}_u\Big((X_i)_{i\in[0,k]} \in A, \widetilde{H}_K > H_{\partial n} = k\Big)\,\mathbb{P}_u\big((X_i)_{i\in[0,m]} \in B, H_{\partial_n} = m\big), \end{aligned}$$

then by reversibility of the walk we see that the above is equal to

$$\frac{\deg(\partial_n)}{\deg(u)} \cdot \mathbb{P}_{\partial_n}\Big((X_i)_{i\in[0,k]} \in A^{\leftarrow}, X_{H_K} = u, H_K = k < \widetilde{H}_{\partial_n}\Big) \cdot \mathbb{P}_u\big((X_i)_{i\in[0,m]} \in B, H_{\partial_n} = m\big).$$

This shows that the measure $Q_K^n/Q_K^n(\mathcal{W})$ is the law of a random walk excursion that starts from ∂_n, it is conditioned to hit K and reparameterised so that it hits K for the first time at time 0. As before, one can then construct a measure ν_n so that for all A we have

$$\nu_n(A \cap \mathcal{W}_K^*) = Q_K^n(\pi^{-1}(A)).$$

By definition we see that the measures Q_K^n converge weakly to Q_K, and hence so do the Poisson processes and this concludes the proof of the claim.

It remains to show that the Aldous Broder interlacements algorithm yields a USF. To see this, note that for the finite graphs G_n, we have already proved that Aldous Broder produces an oriented UST. Using excursions from ∂_n to run Aldous Broder, we see that the correctness of this algorithm in the finite setting together with the claim that we proved implies the statement of the theorem. $\square$

Remark 4.32. This dynamic way of sampling the USF has many advantages, especially when it comes to calculating tail probabilities for certain events as we show below.

As an application, we present the proof of Theorem 4.26 due to Hutchcroft.

Proof of Theorem 4.26. As we have already noted in Lemma 4.30, one-endedness is equivalent to every vertex having a finite past. For every $u \in \mathbb{Z}^d$, every $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we let $\mathrm{past}_t(u,n)$ be the set of vertices in the past of u that have a path of length n to u and they have been generated by $\mathrm{AB}_t(\mathcal{I})$.

Let $\varepsilon > 0$ to be determined. We then have

$$\mathbb{P}(\mathrm{past}_0(0,n) \neq \emptyset) \leq \mathbb{P}(\tau_0(0) \leq \varepsilon) + \mathbb{P}(\tau_0(0) > \varepsilon, \mathrm{past}_0(0,n) \neq \emptyset)\,.$$

We now notice that the event $\{\tau_0(0) > \varepsilon, \mathrm{past}_0(0,n) \neq \emptyset\}$ implies that there exists $u \in \mathrm{past}_\varepsilon(0,n)$ such that the path from 0 to u is not hit by $\mathcal{I}$ in $[0,\varepsilon]$. Indeed, notice that if $u \in \mathrm{past}_0(0,n)$, it means that $\tau_0(u) \geq \tau_0(0)$, and since $\tau_0(0) > \varepsilon$, it follows that $\tau_0(u) > \varepsilon$, i.e. the first trajectory that hits u arrived after time ε. Therefore, $u \in \mathrm{past}_\varepsilon(0,n)$ and the path from 0 to u was not hit by $\mathcal{I}$ during $[0,\varepsilon]$. Indeed, if it were hit, then that would disconnect u from 0. We thus have

$$\begin{aligned}\mathbb{P}(\tau_0(0) > \varepsilon, \mathrm{past}_0(0,n) \neq \emptyset) &\leq \mathbb{P}(\exists\, u \in \mathrm{past}_\varepsilon(0,n) :\ \text{path from 0 to } u \text{ is not hit by } \mathcal{I} \text{ in } [0,\varepsilon]) \\ &\leq \mathbb{E}\left[\sum_{u \in \mathrm{past}_\varepsilon(0,n)} \exp(-\varepsilon \cdot \mathrm{Cap}(\mathrm{path}[0,u]))\right].\end{aligned}$$

When $u \in \mathrm{past}_\varepsilon(0,n)$, the path from 0 to u in the past has length n. To lower bound the capacity of this path, we use the bound

$$\mathrm{Cap}(A) \gtrsim |A|^{(d-2)/d},$$

and hence this gives

$$\mathbb{P}(\tau_0(0) > \varepsilon, \mathrm{past}_0(0,n) \neq \emptyset) \leq \exp(-\varepsilon n^{(d-2)/d}).$$

By symmetry we get

$$\mathbb{E}[|\mathrm{past}_\varepsilon(0,n)|] = \sum_{u \in \mathbb{Z}^d} \mathbb{P}(u \in \mathrm{past}_0(0,n)) = \sum_{u \in \mathbb{Z}^d} \mathbb{P}(0 \in \mathrm{past}_0(u,n)) = \sum_{u \in \mathbb{Z}^d} \mathbb{P}(u \in \mathrm{future}(0,n)) = 1,$$

since there is exactly one vertex in the future at distance n from 0. The probability that 0 is hit in $[0,\varepsilon]$ by a trajectory of random interlacements is given by

$$1 - \exp(-\varepsilon \mathrm{Cap}(0)) \asymp \varepsilon.$$

Taking $\varepsilon = \log n / n^{(d-2)/d}$ shows that

$$\mathbb{P}(\mathrm{past}_0(0,n) \neq \emptyset) \lesssim \frac{\log n}{n^{(d-2)/d}},$$

and hence taking the limit as $n \to \infty$ concludes the proof. $\square$

Theorem 4.33 (Hutchcroft)**.** *For all $d \geq 5$ we have*

$$\mathbb{P}(\mathrm{rad}_{\mathrm{int}}(\mathrm{past}(0)) \geq n) \asymp \frac{1}{n}.$$

Theorem 4.34 (Hutchcroft and Sousi)**.** *For $d = 4$ we have*

$$\mathbb{P}(\mathrm{rad}_{\mathrm{int}}(\mathrm{past}(0)) \geq n) \asymp \frac{(\log n)^{1/3}}{n}.$$

Sketch of proof of lower bound. To prove the lower bound we need to find a strategy to create a big past.

Let $\varepsilon > 0$ to be determined. Let A be the event that 0 is hit by a unique trajectory W of RI in the time interval $[0, \varepsilon]$. Let $X = \mathrm{AB}(W\mid_{[0,\infty)})$. Let η be the infinite path in X. Then it is not hard to check that η has the law of a loop erased random walk in $\mathbb{Z}^4$.

We also define B as follows

$$B = \{W\mid_{(-\infty,0)} \cap \eta[0,n] = \emptyset, \eta[0,n] \text{ is not hit by any other traj. of RI in } [0,\varepsilon]\}.$$

With these definitions it follows that $A \cap B$ implies that the past has intrinsic radius at least n. We then have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,.$$

By the Poisson property of RI, we get that

$$\mathbb{P}(A) = \varepsilon \cdot \mathrm{Cap}(\{0\}) \cdot e^{-\varepsilon \mathrm{Cap}(\{0\})}$$

By the splitting property of Poisson processes we also get

$$\mathbb{P}(B \mid A) = \mathbb{E}\big[\mathbf{1}(W\mid_{(-\infty,0)} \cap \eta[0,n] = \emptyset) \cdot \exp\left(-\varepsilon \mathrm{Cap}(\eta[0,n])\right)\big]\,.$$

We now replace $\mathrm{Cap}(\eta[0,n])$ by its typical value which is $n/(\log n)^{2/3}$ and we take $\varepsilon = (\log n)^{2/3}/n$ to get

$$\mathbb{P}(B \mid A) \gtrsim \mathbb{P}\big(W\mid_{(-\infty,0)} \cap \eta[0,n] = \emptyset\big)\,.$$

Using Lawler's non-intersection exponent between a SRW and the first n steps of a LERW in $\mathbb{Z}^4$ we get

$$\mathbb{P}(B \mid A) \gtrsim \frac{1}{(\log n)^{1/3}}.$$

Putting everything together we obtain

$$\mathbb{P}(A \cap B) \gtrsim \frac{(\log n)^{1/3}}{n}$$

and this concludes the proof. $\square$

Lawler showed that the first n steps of a LERW in $\mathbb{Z}^4$ are produced by loop-erasing the first $n(\log n)^{1/3}$ steps of a SRW. Using the result by Lyons, Peres and Schramm that intersection probabilities between two SRW's are equivalent to intersection probabilities between one walk and the LE of the other one, we see that

$$\begin{aligned}\mathbb{E}[\mathrm{Cap}(\eta[0,n])] \asymp \mathbb{E}\Big[\mathrm{Cap}(\mathrm{LE}(X[0,n(\log n)^{1/3}))\Big] &\asymp \mathbb{E}\Big[\mathrm{Cap}(X[0,n(\log n)^{1/3})\Big] \\ &\asymp \frac{n(\log n)^{1/3}}{\log n} = \frac{n}{(\log n)^{2/3}}.\end{aligned}$$

5 Branching random walks

5.1 Critical trees conditioned to survive

Let $\mu = (\mu_k)_{k\geq 0}$ be a probability distribution on $\mathbb{N}$ with mean 1 and finite variance σ^2. Let $\mathcal{T}_c$ be a Galton Watson tree with offspring distribution μ. We denote its root by $\emptyset$. For every $d \geq 1$ we attach i.i.d. increments of a $\mathbb{Z}^d$ simple random walks to the edges of the tree. For each $x \in \mathbb{Z}^d$ we now define a branching random walk on $\mathbb{Z}^d$ started from x, by assigning to each vertex $u \in \mathcal{T}_c$ the sum of the increments along the edges on the shortest path joining u to $\emptyset$ and translating everything by x. We write $(S_v^x)_{v\in\mathcal{T}_c}$ for the branching random walk started from x.

The local time at $y \in \mathbb{Z}^d$ is defined to be

$$\ell_{\mathcal{T}_c}(y) = \sum_{u\in\mathcal{T}_c} \mathbf{1}(S_u^x = y).$$

Writing Z_n for the number of individuals in generation n and using the independence between the tree and the increments we get that the first moment of the local time is given by

$$\mathbb{E}[\ell_{\mathcal{T}_c}(y)] = \sum_{n=0}^{\infty} \mathbb{E}[Z_n]\, \mathbb{P}_x(X_n = y) = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y) = g(x,y),$$

where X denotes a simple random walk on $\mathbb{Z}^d$ and g stands for the Green's function. Thus we see the critical branching random walk has the same Green's function as a simple random walk.

Suppose we condition the tree to reach generation n. Each particle performs a simple random walk run for n steps and hence covers of the order n points of the ball of radius $\sqrt{n}$ in $\mathbb{Z}^d$. As we will show below, under this conditioning, there will be order n particles that walk in the ball of radius $\sqrt{n}$, and hence in total n^2 points of the ball of radius $\sqrt{n}$ will be covered. The volume of the ball of this radius is of order $n^{d/2}$. We thus see that if $d = 4$, then this ball is filled by a BRW conditioned to be large. Hence we see that dimension 4 is the critical dimension for recurrence in this setup.

We now start with a result of Kolmogorov and Yaglom and a later version by Kesten, Ney and Spitzer (only assuming second moments). Here we present a proof due to Geiger [6].

Theorem 5.1 (Kolmogorov, Yaglom, Kesten, Ney, Spitzer)**.** *Let $(\mu(k))_{k\geq 0}$ be an offspring distribution with mean 1 and finite variance σ^2. If Z_n denotes the size of the n-th generation, we then have*

$$n \cdot \mathbb{P}(Z_n > 0) \to \frac{2}{\sigma^2} \text{ as } n \to \infty.$$

We let R_{n+1} be the index of the leftmost child of the root who has a descendant in generation $n+1$. Formally, let $(T^{(i)})_{i\leq Z_1}$ be the subtrees rooted at the Z_1 children of the root. We write $Z_n(T)$ to denote the size of the n-th generation of the tree T. We then define for $n \geq 0$

$$R_{n+1} = \min\{1 \leq j \leq Z_1 :\; Z_n(T^{(j)}) > 0\}$$

with the usual convention that $\min \emptyset = \infty$.

Lemma 5.2. *For all $i \in \{1, \dots, Z_1\}$ we have*

$$\mathcal{L}(T^{(i)} \mid R_{n+1} = j, Z_1 = k) = \begin{cases} \mathcal{L}(T \mid Z_n = 0) & \text{if } 1 \leq i \leq j-1, \\ \mathcal{L}(T \mid Z_n > 0) & \text{if } i = j, \\ \mathcal{L}(T) & \text{if } j+1 \leq i \leq k, \end{cases}$$

where T stands for a Galton Watson tree with offspring distribution $(\mu(k))$. Moreover, for the joint distribution of (R_{n+1}, Z_1) conditional on $Z_{n+1} > 0$ we have

$$\mathbb{P}(R_{n+1} = j, Z_1 = k \mid Z_{n+1} > 0) = \frac{\mathbb{P}(Z_n > 0)}{\mathbb{P}(Z_{n+1} > 0)} \cdot \mu(k) \cdot \mathbb{P}(Z_n = 0)^{j-1}\,.$$

Proof. To prove the first claim we let (A_i) be measurable sets in the space of rooted planar trees such that for $i \le j-1$ the sets A_i contain trees that do not survive to generation n, while for $i = j$ they do. We then have

$$\mathbb{P}\Big(R_{n+1} = j, Z_1 = k, T^{(i)} \in A_i,\ \forall\, i \le k\Big) = \mu(k) \cdot \prod_{i=1}^{k} \mathbb{P}(T \in A_i)\,.$$

Therefore, we immediately get from this

$$\mathbb{P}(R_{n+1} = j, Z_1 = k, Z_{n+1} > 0) = \mu(k) \cdot \mathbb{P}(Z_n > 0) \cdot \mathbb{P}(Z_n = 0)^{j-1}\,,$$

and hence, this proves the second claim. $\square$

Remark 5.3. Taking the sum over all $k \ge 1, j \le k$ in the above equation we obtain

$$\mathbb{P}(Z_{n+1} > 0) = \mathbb{P}(Z_n > 0) \cdot \sum_{j=1}^{\infty} \mathbb{P}(Z_n = 0)^{j-1} \sum_{k \ge j} \mu(k). \tag{5.1}$$

From this and the fact that $\sum k\mu(k) = 1$ it follows that

$$\frac{\mathbb{P}(Z_{n+1} > 0)}{\mathbb{P}(Z_n > 0)} \to 1 \text{ as } n \to \infty. \tag{5.2}$$

From the above we also get that as $n \to \infty$ the distribution of Z_1 conditional on $Z_n > 0$ converges to the size-biased distribution of μ, i.e. the distribution $\mu_{\mathrm{sb}}(k) = k\mu(k)$ for all $k \ge 0$. Also given Z_1 we have that R_{n+1} has the uniform distribution on $[0, Z_1]$.

We can now describe a construction of a critical tree conditioned on survival up to generation n.

Let $(V_{n+1}, Y_{n+1})_{n \ge 0}$ be independent random variables with distribution

$$\mathbb{P}(V_{n+1} = j, Y_{n+1} = k) = \mu(k) \cdot \frac{\mathbb{P}(Z_n > 0)}{\mathbb{P}(Z_{n+1} > 0)} \cdot \mathbb{P}(Z_n = 0)^{j-1}\,. \tag{5.3}$$

Let $\widetilde{T}_0$ be a GW tree with offspring distribution $(p_k)_{k \ge 0}$ independent of the sequence $(V_{n+1}, Y_{n+1})_{n \ge 0}$. We now inductively construct $\widetilde{T}_{n+1}$ for $n \ge 0$ as follows:

1. The first generation of $\widetilde{T}_{n+1}$ is taken to be Y_{n+1}.
2. To the V_{n+1} individual of the first generation we attach the tree $\widetilde{T}_n$.
3. We attach critical GW trees conditioned on extinction at generation n to the $V_{n+1} - 1$ children to the left of the V_{n+1} individual of the first generation.
4. We attach critical GW trees to the $Y_{n+1} - V_{n+1}$ individuals of the first generation to the right of the V_{n+1} individual.

By induction on n we obtain the following

Proposition 5.4. *For all $n \geq 0$ we have*

$$\mathcal{L}(\widetilde{T}_n) = \mathcal{L}(T \mid Z_n > 0).$$

We now want to find a recursion for the number of individuals $\widetilde{Z}_n$ in the n-th generation of $\widetilde{T}_n$. By construction we see that $\widetilde{Z}_0 = 1$. We now let $(Z_{k,i})_{k,i}$ be independent random variables with $Z_{k,i}$ having the law of the k-th generation size of a critical GW tree for all i. Writing $X_{m+1} = Y_{m+1} - V_{m+1}$ we have

$$\widetilde{Z}_{n+1} = \widetilde{Z}_n + \sum_{i=1}^{X_{n+1}} Z_{n,i}. \tag{5.4}$$

Then it follows immediately from Proposition 5.4 and the construction of $\widetilde{T}_n$

Corollary 5.5. *For all $n \geq 0$ we have*

$$\mathcal{L}(\widetilde{Z}_n) = \mathcal{L}(Z_n \mid Z_n > 0).$$

From (5.3) we get for all $k \geq 0$

$$\mathbb{P}(X_{n+1} = k) = \frac{\mathbb{P}(Z_n > 0)}{\mathbb{P}(Z_{n+1} > 0)} \cdot \sum_{j=k+1}^{\infty} \mu(j) \cdot \mathbb{P}(Z_n = 0)^{j-(k+1)}.$$

Using (5.2) and that by criticality of the tree $\mathbb{P}(Z_n = 0) \to 1$ as $n \to \infty$ we obtain

$$\mathbb{P}(X_n = k) \to \sum_{j \geq k+1} \mu(j) \text{ as } n \to \infty. \tag{5.5}$$

Proof of Theorem 5.1. First of all we have

$$\mathbb{E}[Z_n \mid Z_n > 0] = \frac{1}{\mathbb{P}(Z_n > 0)},$$

since $\mathbb{E}[Z_n] = 1$ for all n by criticality. By Corollary 5.5 we have

$$\mathbb{E}[Z_n \mid Z_n > 0] = \mathbb{E}\Big[\widetilde{Z}_n\Big] = 1 + \sum_{k=1}^{n} \mathbb{E}\Big[\widetilde{Z}_k - \widetilde{Z}_{k-1}\Big].$$

For every $k \geq 1$ we have by (5.4)

$$\mathbb{E}\Big[\widetilde{Z}_k - \widetilde{Z}_{k-1}\Big] = \mathbb{E}\left[\sum_{i=1}^{X_k} Z_{k-1,i}\right] = \mathbb{E}[X_k],$$

where we used that $\mathbb{E}[Z_{k,i}] = 1$ for all k, i. Using that $\mathbb{E}[X_n] \to \sum_k k \sum_{j \geq k+1} \mu(j)$ as $n \to \infty$ we deduce

$$\sum_{i=1}^{n} \frac{\mathbb{E}[X_i]}{n} \to \sum_k k \sum_{j \geq k+1} \mu(j) \text{ as } n \to \infty.$$

This last sum is equal to

$$\sum_k k \sum_{j \geq k+1} \mu(j) = \sum_{j=1}^{\infty} \sum_{k \leq j-1} k\mu(j) = \sum_{j=1}^{\infty} \frac{j(j-1)}{2} \cdot \mu(j) = \frac{\sigma^2}{2}$$

and this concludes the proof. $\square$

Corollary 5.6. *Let Z_n be the n-th generation size of a critical GW tree. Then we have as $n \to \infty$*

$$\mathbb{E}[Z_n \mid Z_n > 0] \asymp n^2.$$

As $n \to \infty$ the tree conditioned on $Z_n > 0$ we saw that it converges to a tree where the first individual produces offspring according to the size-biased distribution and then a uniformly chosen one is the special vertex that is going to produce offspring according to the size-biased distribution, while the rest will produce offspring according to μ. Every other vertex except for the special ones produce offspring according to μ. This is called Kesten's tree and it consists of a semi-infinite line that we call the spine and finite trees to the right and the left of the spine.

We now consider Kesten's tree. Let L_n be the number of children to the left of the distinguished vertex of generation n and R_n the number of vertices to its right. Then we get

$$\mathbb{P}(L_n = i, R_n = j) = \frac{(i+j+1)\mu(i+j+1)}{i+j+1} = \mu(i+j+1),$$

and hence taking the sum over all j of the above we obtain

$$\mathbb{P}(L_n = i) = \sum_{j \geq 0} \mu(i+j+1)$$

and similarly for the right one.

Example 5.7. Let T be Kesten's tree. Let t be an infinite tree and let v_0 be a spine vertex of v_0 which is not the root. Then

$$\begin{aligned} &\mathbb{P}(T \text{ is equal to } t \text{ up to } v_0) = \deg(\emptyset) \cdot \mu(\deg(\emptyset)) \cdot \frac{1}{\deg(\emptyset)} \\ &\times \prod_{\substack{v \in \text{ spine of } t \\ \text{before } v_0}} (\deg(v) - 1) \cdot \mu(\deg(v) - 1) \cdot \frac{1}{\deg(v) - 1} \cdot \prod_{\substack{v \notin \text{ spine of } t \\ \text{up to } v_0}} \mu(\deg(v) - 1) \\ &= \mu(\deg(\emptyset)) \prod_{v \in t \text{ up to } v_0} \mu(\deg(v) - 1). \end{aligned} \tag{5.6}$$

We thus see that the distribution of T depends on the particular choice of root.

To get a tree which is invariant with respect to the choice of the root we simply need to change the root offspring distribution and how we choose the special vertex among its offspring.

Definition 5.8. The infinite invariant tree is defined as follows: the spine is a semi-infinite line of nodes $(\emptyset, u_1, \ldots)$. Each spine node u_i for $i > 0$ produces a random number of offspring Z_i according to the size-biased distribution $\mu_{\mathrm{sb}}(k) = k\mu(k)$ for each $k \geq 0$. A uniformly chosen offspring of u_i is identified with u_{i+1} and in this way it partitions the $Z_i - 1$ remaining offspring to left and right children. To each of these in turn we attach a critical Galton Watson tree with offspring distribution μ. The root produces Z_0 offspring distributed according to $\mu'(i) = \mu(i-1)$ for $i \geq 1$. The first of its offspring is identified with u_1 and the remaining $Z_0 - 1$ produce critical GW trees with offspring distribution μ. We denote this infinite tree by $\mathcal{T}$. We define the past $\mathcal{T}_-$ to be the set of vertices to the left of the spine and the future $\mathcal{T}_+$ the ones to the right. We label the vertices in the future using depth first search from the root. We label the vertices in the past in order with negative labels using depth first search from infinity.

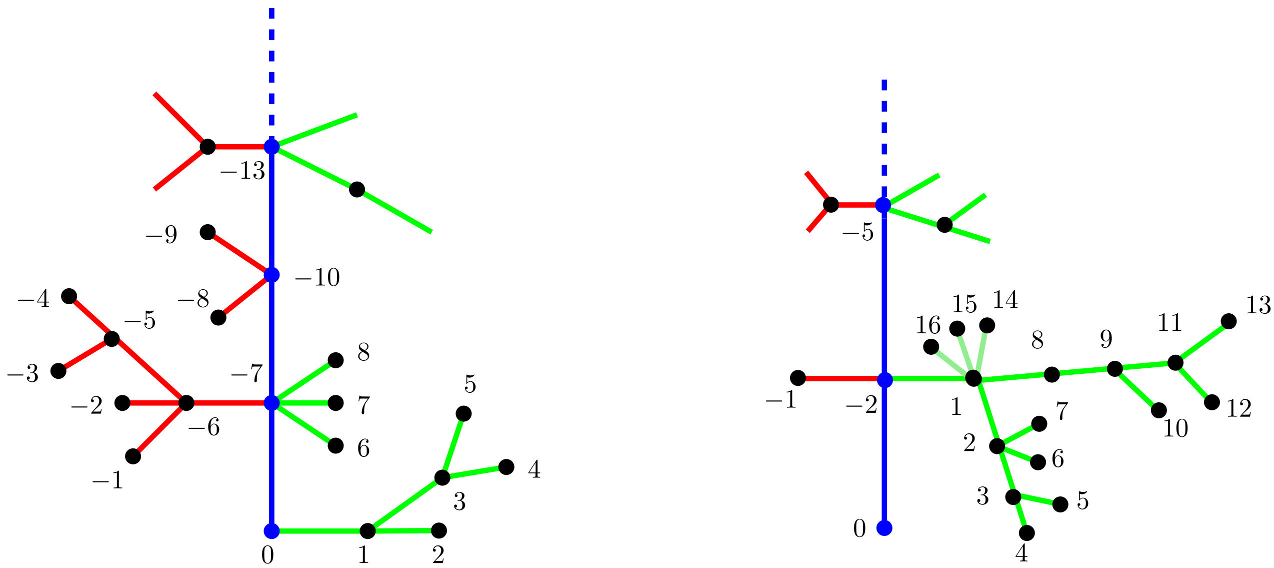

Figure 1: In both trees the future is the green part, while the past consists of the red and the blue parts. To obtain the second tree we shifted all labels of the first tree by 8 and rooted it at the vertex with label 0.

Remark 5.9. From (5.6) we see that the infinite invariant tree is invariant with respect to the choice of the root. In particular, if we shift the labels of all vertices by 1, then it will have exactly the same distribution. See also [12] and [13].

Finally we add the superscript x to a tree, i.e. we write $\mathcal{T}_c^x$ (resp. $\mathcal{T}^x, \mathcal{T}_-^x, \mathcal{T}_+^x$) to denote the range of the branching random walk when the root vertex starts from $x \in \mathbb{Z}^d$.

We now calculate the local time for the past of the infinite invariant tree. For $y \in \mathbb{Z}^d$ we define the local time at y in the past as follows

$$\ell_{\mathcal{T}_-^x}(y) = \sum_{u \in \mathcal{T}_-} \mathbf{1}(S_u^x = y).$$

Check that

$$\mathbb{E}\Big[\ell_{\mathcal{T}_-^x}(y)\Big] = \frac{\sigma^2}{2} \cdot \sum_{z \in \mathbb{Z}^d} g(x-z)g(z-y) + O(g(x,y)) = \frac{\sigma^2}{2} \cdot g * g(x-y) + O(g(x,y)).$$

Exercise 5.10. Show that for all $x \in \mathbb{Z}^d$ we have

$$g * g(x) \asymp \frac{1}{\|x\|^{d-4}+1}.$$

5.2 Hitting probabilities

In this section we follow closely [25].

Let $\mathcal{T}_c^x$ be a critical BRW started from x. Let $K \subseteq \mathbb{Z}^d$ be a finite set. We want to calculate the probability that the range of the tree intersects the set K. To define the first hitting vertex of K

we use depth first search from the root. Suppose that $(v_0, \ldots, v_k)$ be the unique simple path in $\mathcal{T}_c$ from the root $\emptyset$ to the first hitting vertex in K. Let $\Gamma = (S^x_{v_0}, \ldots, S^x_{v_k})$ be the path that visits K for the first time. For every $i = 0, \ldots, k-1$ we let $\widetilde{a}_i$ be the number of children to the left of v_i (i.e. older than v_i) and $\widetilde{b}_i$ the number of children to the right of v_i (i.e. younger than v_i).

Let $\gamma = (\gamma(0), \ldots, \gamma(k))$ be a nearest neighbour path in $\mathbb{Z}^d$ with $\gamma(0) = x$, $\gamma(k) \in K$ and $\gamma(j) \notin K$ for all $j \le k-1$. For a path γ we write $s(\gamma)$ for the probability that a SRW followed this path, i.e. $s(\gamma) = (2d)^{-|\gamma|}$. We then have for all $m_i, \ell_i \in \mathbb{N}$ for $i \le k-1$

$$\mathbb{P}\Big(\Gamma = \gamma, \widetilde{a}_i = \ell_i, \widetilde{b}_i = m_i, \forall\, i = 0, \ldots, k-1\Big) = s(\gamma) \prod_{i=0}^{k-1} \mu(\ell_i + m_i + 1)\; \mathbb{P}\Big(\widehat{\mathcal{T}}_c^{\gamma(i)} \cap K = \emptyset\Big)^{\ell_i}, \tag{5.7}$$

where we denote by $\widehat{\mathcal{T}}_c$ a GW tree with offspring distribution μ conditioned on the root having exactly one child, i.e. for all $x \in \mathbb{Z}^d$ we have

$$\mathbb{P}\Big(\widehat{\mathcal{T}}_c^x \cap K = \emptyset\Big) = \mathbb{P}(\mathcal{T}_c^x \cap K = \emptyset \mid Z_1 = 1)\,.$$

If we sum up (5.7) over all m_i and ℓ_i, for $i \le k-1$, we then get

$$\begin{aligned}\mathbb{P}(\Gamma = \gamma) &= \sum_{\substack{\ell_1,\ldots,\ell_k\\ m_1,\ldots,m_k}} \mathbb{P}\Big(\Gamma = \gamma, \widetilde{a}_i = \ell_i, \widetilde{b}_i = m_i,\ \forall\, i = 1, \ldots, k\Big)\\ &= s(\gamma) \prod_{i=0}^{k-1} \sum_{\ell_i, m_i \in \mathbb{N}} \mu(\ell_i + m_i + 1)\mathbb{P}\Big(\widehat{\mathcal{T}}_c^{\gamma(i)} \cap K = \emptyset\Big)^{\ell_i}.\end{aligned} \tag{5.8}$$

We now introduce another probability measure $\widetilde{\mu}$ which is called the adjoint measure of μ defined as follows

$$\widetilde{\mu}(i) = \sum_{j \ge 0} \mu(i + j + 1).$$

Indeed, this is a probability measure, because μ has mean 1. Using this definition in (5.8) we get

$$\mathbb{P}(\Gamma = \gamma) = s(\gamma) \prod_{i=0}^{k-1} \sum_{\ell_i \in \mathbb{N}} \widetilde{\mu}(\ell_i)\mathbb{P}\Big(\widehat{\mathcal{T}}_c^{\gamma(i)} \cap K = \emptyset\Big)^{\ell_i}.$$

Definition 5.11. We define an adjoint tree, by simply changing the offspring distribution of the root to be $\widetilde{\mu}$ and the rest according to μ. The measure $\widetilde{\mu}$ is defined by

$$\widetilde{\mu}(i) = \sum_{j \ge i+1} \mu(j)$$

and is called the adjoint measure of μ. We write $\widetilde{\mathcal{T}}_c$ for the adjoint tree.

With the above definition, we immediately see that

$$\sum_{\ell_i \in \mathbb{N}} \widetilde{\mu}(\ell_i)\mathbb{P}\Big(\widehat{\mathcal{T}}_c^{\gamma(i)} \cap K = \emptyset\Big)^{\ell_i} = \mathbb{P}\Big(\widetilde{\mathcal{T}}_c^{\gamma(i)} \cap K = \emptyset\Big).$$

For every $z \in \mathbb{Z}^d$ we define

$$r(z) = \mathbb{P}\Big(\widetilde{\mathcal{T}}_c^z \cap K \neq \emptyset\Big).$$

With this notation we then obtain

$$\mathbb{P}(\Gamma = \gamma) = s(\gamma) \cdot \prod_{i=0}^{k-1} (1 - r(\gamma(i))) =: b_K(\gamma). \tag{5.9}$$

What we see from the above formula is that the distribution of Γ is this of a simple random walk which is killed at location x with probability $r(x)$.

Lemma 5.12. *Let $d \geq 5$. We show that for $R > 0$ and $\|x\|$ large we have*

$$\mathbb{P}(\mathcal{T}_c^x \cap B(0,R) \neq \emptyset) \asymp g(x) \cdot R^{d-4}.$$

Proof. Recall that we take the first hitting time of a set by a BRW to be the first vertex in the depth first search order whose location hits the set. Using (5.9) we get

$$\begin{aligned}
\mathbb{P}(\mathcal{T}_c^x \cap B(0,R) \neq \emptyset) &= \sum_{\gamma : x \to \partial B(0,R)} b_{B(0,R)}(\gamma) \\
&= \sum_{b \in \partial B(0,2R)} \sum_{\substack{\gamma_1 : x \to b \\ \gamma_1 \subseteq B(0,2R)^c}} b_{B(0,R)}(\gamma_1) \sum_{\gamma_2 : b \to \partial B(0,2R)} b_{B(0,R)}(\gamma_2) \\
&\leq \mathbb{P}_x(X \text{ hits } B(0,2R)) \cdot \sup_{b \in \partial B(0,2R)} \mathbb{P}\Big(\mathcal{T}_c^b \cap B(0,R) \neq \emptyset\Big),
\end{aligned}$$

where we simply bounded from above $b(\gamma)$ by $s(\gamma)$ and X denotes a simple random walk in $\mathbb{Z}^d$. We now get

$$\mathbb{P}_x(X \text{ hits } B(0,2R)) \asymp g(x) \cdot R^{d-2}.$$

Writing Z_k for the size of the k-th generation of the tree $\mathcal{T}_c$ we have for all $b \in \partial B(0,2R)$

$$\mathbb{P}\Big(\mathcal{T}_c^b \cap B(0,R) \neq \emptyset\Big) \leq \sum_{k=1}^{\lfloor R^2 \rfloor} \mathbb{P}\Big(Z_k \neq 0, Z_{k+1} = 0, \mathcal{T}_c^b \cap B(0,r) \neq \emptyset\Big) + \mathbb{P}\big(Z_{\lfloor R^2 \rfloor} \neq 0\big).$$

The second term is upper bounded by $1/R^2$ by Kolmogorov's result. So we now turn to the sum appearing above. By a union bound over all individuals of the k-th generation together with the fact that every particle performs a simple random walk we get

$$\begin{aligned}
\mathbb{P}\Big(Z_k \neq 0, Z_{k+1} = 0, \mathcal{T}_c^b \cap B(0,r) \neq \emptyset\Big) &\leq \mathbb{E}\big[Z_k (\mu(0))^{Z_k}\big]\, \mathbb{P}_x\big(H_{B(0,R)} \leq k\big) \\
&\leq \mathbb{E}\big[Z_k (\mu(0))^{Z_k}\big] \exp(-cR^2/k),
\end{aligned}$$

where H_A stands for the first hitting time of the set A by a simple random walk and c is a positive constant. Since μ has mean 1, we get that $\mu(0) < 1$, and hence using again Kolmogorov's result we get

$$\mathbb{E}\big[Z_k (\mu(0))^{Z_k}\big] \leq \mathbb{P}(Z_k > 0) \lesssim \frac{1}{k}.$$

Putting everything together shows that

$$\mathbb{P}\Big(\mathcal{T}_c^b \cap B(0,R) \neq \emptyset\Big) \lesssim \frac{1}{R^2},$$

and hence this concludes the proof of the upper bound, i.e. that

$$\mathbb{P}(\mathcal{T}_c^x \cap B(0,R) \neq \emptyset) \lesssim g(x) \cdot R^{d-4}.$$

We turn to the lower bound. We denote by η_R^x the vertices of the tree that are the first ones to reach $\partial B(0,R)$ and none of their ancestors ever touched $\partial B(0,R)$. To prove the lower bound we will use the Payley-Zygmund inequality

$$\mathbb{P}(|\eta_R^x| > 0) \geq \frac{\mathbb{E}[|\eta_R^x|]^2}{\mathbb{E}\left[|\eta_R^x|^2\right]}.$$

We have

$$\mathbb{E}[|\eta_R^x|] = \sum_{n=0}^{\infty} \mathbb{E}[Z_n] \, \mathbb{P}_x\big(H_{B(0,R)} = n\big) = \mathbb{P}_x\big(H_{B(0,R)} < \infty\big) \asymp g(x) \cdot R^{d-2}.$$

For a vertex w of the tree we write ξ_w for its offspring. For the second moment we have

$$\mathbb{E}\left[|\eta_R^x|^2\right] = \mathbb{E}[|\eta_R^x|] + \mathbb{E}\left[\sum_{\substack{u,v\in\mathcal{T}_c \\ u\neq v}} \sum_{w\in\mathcal{T}_c} \mathbf{1}(w \text{ is the MRCA of } u,v)\mathbf{1}(u\in\eta_R)\mathbf{1}(v\in\eta_R)\right].$$

Since u and v do not have any ancestors in η_R^x, it follows that w must have at least two children and u and v must belong to two distinct trees of its offspring. There are $\xi_w(\xi_w - 1)$ ways of choosing which trees u and v will belong to. Summing over all possible generations of w we get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{u,v\in\mathcal{T}_c \\ u\neq v}} \sum_{w\in\mathcal{T}_c} \mathbf{1}(w \text{ is the MRCA of } u,v)\mathbf{1}(u\in\eta_R)\mathbf{1}(v\in\eta_R)\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\sum_{w\in\mathcal{T}_c} \mathbf{1}(|w|=k)\mathbf{1}(S_w^x \notin B(0,R))\xi_w(\xi_w-1)(\mathbb{P}_{S_w^x}\big(H_{B(0,R)}<\infty\big))^2\right] \\
&= \sigma^2 \sum_{k=0}^{\infty} \mathbb{E}[Z_k] \, \mathbb{E}_x\left[\mathbf{1}(X_k \notin B(0,R))(\mathbb{P}_{X_k}\big(H_{B(0,R)}<\infty\big))^2\right] \\
&\lesssim R^{2(d-2)} \cdot \sum_{k=0}^{\infty} \mathbb{E}_x\left[\frac{\mathbf{1}(X_k \notin B(0,R))}{\|X_k\|^{2(d-2)}}\right] = R^{2(d-2)} \cdot \sum_{z\notin B(0,R)} g(x-z) \cdot \frac{1}{\|z\|^{2(d-2)}}.
\end{aligned}$$

We now see that this last expression is equal to

$$R^{2(d-2)} \cdot \sum_{z\notin B(0,R)} g(x-z) \cdot \frac{1}{\|z\|^{2(d-2)}} \lesssim \begin{cases} \frac{R^d}{\|x\|^{d-2}} & \text{if } d \geq 5 \\ \frac{R^4}{\|x\|^2} \cdot \log(1+\|x\|/R) & \text{if } d = 4. \end{cases}$$

This now completes the proof. $\square$

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