POLYNOMIAL TERM STRUCTURE MODELS

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ABSTRACT. In this article, we explore a class of tractable interest rate models that have the property that the prices of zero-coupon bonds can be expressed as polynomials of a state diffusion process. These models are, in a sense, generalisations of exponential polynomial models. Our main result is a classification of such models in the spirit of Filipovic's maximal degree theorem for exponential polynomial models.

1. INTRODUCTION

A factor model of the interest rate term structure is one in which the time-t spot interest rate is of the form

$$r_t = R(Z_t)$$

and the time-t price of a bond of maturity T is of the form

$$P_t(T) = H(T-t, Z_t)$$

where $R : \mathbb{R}^d \to \mathbb{R}$ and $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ are given functions and $Z = (Z_t)_{t \geq 0}$ is an d-dimensional factor process. Here we consider only bonds which pay no coupons, suffer no default risk, and have unit face value. To match the terminal price, we assume that

$$H(0,z) = 1 \text{ for all } z. \tag{1}$$

More importantly, to ensure that there is no arbitrage, we assume the existence of a probability measure $\mathbb{Q}$ under which the discounted bond prices, defined by

$$\tilde{P}_t(T) = e^{-\int_0^t r_s ds} P_t(T),$$

are local martingales. Of course, this assumption imposes a constraint on the functions R and H and the dynamics of Z under $\mathbb{Q}$. Indeed, in the case $d = 1$, if Z is assumed to be a solution of the stochastic differential equation

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dW_t, \tag{2}$$

where W is a scalar Brownian motion and b and σ are given functions, then Itô's formula yields the appropriate consistency condition

$$\partial_x H = b\, \partial_z H + \frac{1}{2}\sigma^2\, \partial_{zz} H - RH \text{ for all } (x,z) \in (0,\infty) \times I \tag{3}$$

where $I \subseteq \mathbb{R}$ is the state space of the process Z. In principle, the above partial differential equation (3) with boundary condition (1) can be solved numerically whenever the functions b, σ and R are suitably well-behaved. However, to actually implement such a model, one must first calibrate the parameters, and unfortunately, resorting to a numerical methods at this stage can obscure the relationship between the dynamics of the factor process and the resulting bond prices. Therefore, there has been considerable interest in developing tractable models, where the function H is of reasonably explicit form.

Perhaps the two most famous tractable factor models are those of Vasicek [12] and Cox, Ingersoll & Ross [2]. In these models the factor process is identified with the spot interest rate, so in the notation above, $R(z) = z$, the functions b and σ^2 are assumed to be affine, and the function H is of the exponential affine form

$$H(x,z) = e^{h_0(x) + h_1(x) z}.$$

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It is easy to see that the consistency equation (3) reduces to a system of coupled Riccati ordinary differential equations for the functions h_0 and h_1 with boundary conditions $h_0(0) = h_1(0) = 0$. Duffie & Kan [5] studied exponential affine models where the factor process is of arbitrary dimension $d \geq 1$, leading to much study of the properties of these models by a number of researchers. A notable contribution to this literature is the general characterisation of exponential affine term structure models by Duffie, Filipovic & Schachermayer [4].

An exponential affine model can be considered a special case of the family of exponential quadratic models. An early example of a quadratic model was proposed by Longstaff [10], and has since been developed and generalised by Jamshidian [8], Leippold & Wu [9], and Chen, Filipovic & Poor [1] among others.

One may wonder if there exist non-trivial exponential polynomial models of arbitrary degree. Filipovic answered this question in the negative, by showing that the maximal degree for exponential polynomial models is necessarily two. That is to say, the exponential quadratic models are indeed the most general class of exponential polynomial models.

In this article, we consider the class of polynomial models. In the case where the factor process is scalar-valued, the function H is of the form

$$H(x,z) = \sum_{k=0}^{n} g_k(x) z^k \tag{4}$$

for $n+1$ functions $g_k : \mathbb{R}_+ \to \mathbb{R}$. The main result is a classification of all such models when the factor process is assumed to satisfy an SDE of the form of equation (2). It turns out that the functions b, σ and R are necessarily polynomials of low degree and the functions g_k solve a system of coupled linear ODEs. In light of Filipovic's maximal degree theorem for exponential polynomial models, it might come as a surprise the degree n is not constrained; however, an exponential quadratic model can be seen as the $n \to \infty$ limit, in a certain sense, of a sequence polynomial models.

This work is inspired by the interest rate model of Siegel [11]. He showed that for all integers $d \geq 1$ there exists explicit functions $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $R : \mathbb{R}^d \to \mathbb{R}$ and $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$, depending on d parameters such that if Z is a solution of the stochastic differential equation

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dW_t$$

where W is a d-dimensional Brownian motion,

$$r_t = R(Z_t)$$

and

$$P_t(T) = g(T-t, Z_t),$$

then the processes $\tilde{P}(T)$ are martingales for each $T \geq 0$, where

$$\tilde{P}_t(T) = e^{-\int_0^t r_s ds} P_t(T).$$

Furthermore, the functions a and b are quadratic and the functions R is and $g(x,\cdot)$ are affine for all $x \geq 0$, and $g(0,z) = 1$ for all $z \in \mathbb{R}^d$. In particular, the random variables $P_t(T)$ constitutes an arbitrage-free bond price model, where r_t is the corresponding spot interest rate.

A related work is that of Cuchiero, Keller-Ressel & Teichmann [3], who characterise a class of time-homogeneous Markov process Y with the property that the n-th (mixed) moments can be expressed as a polynomial of the initial point Y_0 of degree at most n. Indeed, consider the $d = 1$ case and let F_n be the family of polynomials of degree at most n:

$$F_n = \left\{ f : f(z) = \sum_{k=0}^{n} f_k z^k,\ f_k \in \mathbb{R} \right\}. \tag{5}$$

They study the processes Y that have the property that for all $t \geq 0$ and for *any* degree n and *any* polynomial $g \in F_n$, there exists a polynomial $h \in F_n$ such that

$$\mathbb{E}[g(Y_t)|Y_0 = y] = h(y).$$

In contrast, in this work we study processes Z such that for all $t \geq 0$ there exists a polynomial $h \in F_n$ such that

$$\mathbb{E}[e^{-\int_0^t R(Z_s)ds}|Z_0 = z] = h(z)$$

where the function R and the degree n are *fixed*. In particular, their results do not imply ours, or vice versa. For further applications of polynomial preserving processes to finance, consult the recent paper of Filipovic and Larsson [7].

In the remainder of this article is arranged as follows. In section 2, we present the main result, a classification of interest rate models in which the bond price can be expressed as a polynomial of a scalar factor process. In section 3, we consider two concrete examples of this class of models and further analyse their properties. Finally in section 4, we briefly discuss two extensions: a Hull–White-type extension where the coefficients are allowed to be time dependent, and the case when the factor has dimension $d > 1$.

2. THE MAIN RESULTS

To more clearly see the structure of the argument we consider only the case $d = 1$ in this section. The multi-dimensional case is considered in section 4. This section contains the main result of this paper, a classification of polynomial term structure models. We fix a degree $n \geq 1$, and let $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be of a polynomial in the second variable as in equation (4). To match the boundary condition (1) we will assume

$$g_0(0) = 1 \text{ and } g_k(0) = 0 \text{ for all } 1 \leq k \leq n. \tag{6}$$

We will assume that the coefficient functions $(g_k)_k$ are linearly independent. We also assume that the scalar factor process Z is a non-explosive solution of the SDE (2) where the state space $I \subseteq \mathbb{R}$ is a non-empty interval.

Theorem 2.1. The function H satisfies the PDE (3) if and only if the following conditions hold true:
Case $n = 1$.

(A) $R(z) = R_0 + R_1 z$ and $b(z) = b_0 + b_1 z + b_2 z^2$ where $R_1 = b_2$.
(B) (g_0, g_1) is the unique solution to the system of linear ODEs

$$\begin{aligned} \dot{g}_0 &= -R_0 g_0 + b_0 g_1 \\ \dot{g}_1 &= -R_1 g_0 + (b_1 - R_0) g_1 \end{aligned}$$

subject to the boundary conditions (6).

Case $n \geq 2$.

(A) $R(z) = R_0 + R_1 z + R_2 z^2$, $b(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3$ and $\sigma^2(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$ where the coefficients are such that

$$R_2 = \tfrac{n}{2} b_3 = -\tfrac{n(n-1)}{2} a_4 \text{ and } R_1 = n b_2 + \tfrac{n(n-1)}{2} a_3.$$

(B) $(g_0, \ldots, g_n)$ is the unique solution to the system of linear ODEs

$$\begin{aligned} \dot{g}_k =& g_{k-2}\left((k-2)b_3 + \frac{(k-2)(k-3)}{2} a_4 - R_2\right) + g_{k-1}\left((k-1)b_2 + \frac{(k-1)(k-2)}{2} a_3 - R_1\right) \\ &+ g_k\left(k b_1 + \frac{k(k-1)}{2} a_2 - R_0\right) + g_{k+1}\left((k+1)b_0 + \frac{k(k+1)}{2} a_1\right) + g_{k+2} \frac{(k+2)(k+1)}{2} a_0 \end{aligned}$$

subject to the boundary conditions (6), where we interpret $g_{-2} = g_{-1} = g_{n+1} = g_{n+2} = 0$.

Before proceeding to the proof, we pause for several remarks.

Remark 1. The solution of the system of ODEs appearing in condition (B) of Theorem 2.1 can be equivalently described as follows. Let $S = (S_{i,j})_{i,j=0}^n$ be the $(n+1) \times (n+1)$ matrix with entries

$$S_{j+k,j} = j b_{k+1} + \tfrac{j(j-1)}{2} a_{k+2} - R_k$$

and where $R_k = b_k = a_k = 0$ when $k < 0$ and $R_k = b_{k+1} = a_{k+2} = 0$ when $k > 2$. For instance, when $n \geq 4$, the matrix has the form

$$S = \begin{pmatrix} -R_0 & b_0 & a_0 & & & \\ -R_1 & b_1 - R_0 & 2b_0 + a_1 & 3a_0 & & \\ -R_2 & b_2 - R_1 & 2b_1 + a_2 - R_0 & 3b_0 + 3a_1 & 6a_0 & \\ & b_3 - R_2 & 2b_2 + a_3 - R_1 & 3b_1 + 3a_2 - R_0 & 4b_0 + 6a_1 & \ddots \\ & & 2b_3 + a_4 - R_2 & 3b_2 + 3a_3 - R_1 & 4b_1 + 6a_2 - R_0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Now letting

$$G(x) = \begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix}.$$

The ODE becomes

$$\dot{G} = SG,$$

and, in particular, the solution can be expressed as

$$G(x) = e^{Sx}G(0),$$

where the boundary condition is given by

$$G(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark 2. Assuming that S has $n+1$ distinct real eigenvalues $\lambda_0, \ldots, \lambda_n$, we know from elementary linear algebra that we can express G via

$$G(x) = \sum_{i=0}^{n} p_i e^{\lambda_i x}$$

for a collection of $n+1$ vectors $(p_i)_i$ in $\mathbb{R}^{n+1}$. Hence the bond pricing function is of the form

$$H(x,z) = \sum_{i=0}^{n} P_i(z) e^{\lambda_i x}$$

where the function

$$P_i(z) = \sum_{k=0}^{n} p_{i,k} z^k$$

is the polynomial whose coefficients are given by the vector p_i. That is to say, the bond price can be seen to be a linear combination of the bond prices arising from models with constant interest rates $r = -\lambda_i$, where the coefficients of the combination depend on the factor process. Also note that generically the long maturity interest rate in this model is given by

$$\lim_{x\to\infty} -\frac{1}{x}\log H(x,z) = -\max_i \lambda_i.$$

Remark 3. Notice that the eigenvalues of the matrix S are the zeros of the characteristic polynomial which has degree $n+1$. It is well know that there exists an explicit formula, discovered by Ferrari in 1540, for the zeros of quartic polynomials, and hence the eigenvalues of S can be expressed in a closed formula in terms of the matrix entries when $n \leq 3$. In particular, in this case, the functions g_k can be written, at least in principle, in terms of the model parameters.

When $n \geq 4$, there is little hope for explicit formulae for the functions g_k in terms of the model parameters. However, note that the matrix is sparse, in the sense that there are at most five non-zero matrix entries per row. In particular, the product of the matrix exponential e^{Sx} and the vector $G(0)$ can be computed efficiently, and hence the lack of explicit formulae is not necessarily a prohibitive disadvantage.

Remark 4. When $n \geq 2$, it seems as though there are ten free parameters: n, R_0, $b_0, \ldots, b_2$, and $a_0, \ldots, a_4$. However, since we are really interested in the interest rate, but not the factor process, and since the function R is quadratic, we need only consider two subclasses of models.

Indeed, if $R_2 = 0$ so that the function R is affine, we can make a change of variables so that $R(z) = z$, and hence the factor Z can be identified with the short rate r. Also note that $b_3 = a_4 = 0$ and hence the SDE for r is of the seven parameter 3/2-type model family

$$dr_t = (b_0 + b_1 r_t + \tfrac{c}{n} r_t^2)dt + \sqrt{a_0 + a_1 r_t + a_2 r_t^2 + \tfrac{2(1-c)}{n(n-1)} r_t^3}\, dW_t.$$

Note that by setting $a_2 = 0$ we see that this polynomial term structure model approximates, in some sense, an exponential affine model when n is large. Note also that the equations $\frac{c}{n} = b_2$ and $\frac{2(1-c)}{n(n-1)} = a_3$ does not uniquely identify the pair (c,n). For instance, the model

$$dr_t = \tfrac{7}{12} r_t (r_t - 2)dt + r_t \sqrt{\tfrac{1}{6}(1 - r_t)} dW_t$$

corresponds to both $(c,n) = (\frac{7}{6}, 2)$ and $(c,n) = (\frac{7}{2}, 6)$. However, in our terminology this model is of degree $n = 2$ and *not* $n = 6$, since the functions $g_3 = g_4 = g_5 = g_6 = 0$ are not linearly independent.

Otherwise, if $R_2 \neq 0$, we can make another change of variables so that $R(z) = R_0 \pm z^2$. The factor process then evolves according to one of the following seven parameter SDEs

$$dZ_t = (b_0 + b_1 Z_t \pm \tfrac{1}{n} Z_t^2 (2Z_t - c))dt + \sqrt{a_0 + a_1 Z_t + a_2 Z_t^2 \mp \tfrac{2}{n(n-1)} Z_t^3 (Z_t - c)} dW_t.$$

Note that in both cases, the parameter n must be an integer greater than one. In particular, calibration of such models to financial data must impose this constraint.

Remark 5. In the case $n = 1$, the affine function R is monotone, so there is no loss of generality taking $R_0 = 0$ and $R_1 = 1$. In this case $r_t = Z_t$ and the short rate model becomes

$$r_t = (b_0 + 2c r_t + r_t^2)dt + \sigma(r_t) dW_t.$$

In this case, the functions g_0 and g_1 can be computed explicitly:

$$g_0(x) = [\cosh(qx) - \tfrac{c}{q} \sinh(qx)] e^{cx}$$

and

$$g_1(x) = -\frac{1}{q} \sinh(qx) e^{cx}$$

where

$$q = \sqrt{c^2 - b_0}.$$

The bond pricing function is then

$$H(x,r) = \frac{1}{2}[1 + (c-q)r](1 + c/q) e^{(c-q)x} + \frac{1}{2}[1 + (c+q)r](1 - c/q) e^{(c+q)x}.$$

This example is a special case of the models developed by Siegel [11]. It is interesting to note this calculation is independent of the function σ. That is to say, set of current bond prices is not sufficient to fully calibrate the model. The parameter σ could, in principle be estimated from historical data. Alternatively it could be calibrated from other interest derivatives.

We are now ready to present the proof of Theorem 2.1.

Proof. Let

$$A_k(z) = k b(z) z^{k-1} + \frac{k(k-1)}{2} \sigma^2(z) z^{k-2} - R(z) z^k.$$

Equation (3) holds if and only if the equation

$$\sum_{k=1}^{n} \dot{g}_k(x) z^k = \sum_{k=1}^{n} g_k(x) A_k(z) \tag{7}$$

holds identically.

We first show that if equation (7) holds then the functions $A_k \in F_n$ for all k, where F_n are the polynomials of degree at most n defined in equation (5). To see this, use the assumed linear independence of the functions $(g_i)_i$ to pick $n+1$ points $0 \leq x_0 < \ldots < x_n$ such that the $(n+1) \times (n+1)$ matrix $(g_i(x_j))_{i,j}$ is invertible. By evaluating equation (7) at the points $(x_j)_j$ and solve for the $A_i(z)$, we see that $A_i(z)$ is a linear combination of monomials z^k of degree at most n.
Case $n = 1$. Note that

$$R(z) = A_0(z)$$
$$b(z) = A_1(z) + z R(z).$$

Since A_0 and A_1 are in F_1, i.e. are affine, then R is affine and b is quadratic. Letting $b(z) = b_0 + b_1 z + b_2 z^2$ and $R(z) = R_0 + R_1 z$ the above system equation implies $b_2 = R_1$. Finally, the identity (7) becomes

$$\dot{g}_0 + \dot{g}_1 z = g_0(R_0 + R_1 z) + g_1(b_0 + (b_1 - zR_0)z).$$

Equating coefficients of z yields the necessity and sufficiency of the system of ODEs.
Case $n \geq 2$. Note that

$$\begin{aligned} R(z) &= A_0(z) \\ b(z) &= A_1(z) + zR(z) \\ \sigma^2(z) &= A_2(z) - 2zb(z) + z^2 R(z). \end{aligned}$$

Since the functions A_i are polynomials, so are the functions R, b, and σ^2. On the other hand

$$\begin{aligned} A_n(z) &= nb(z)z^{n-1} + \frac{n(n-1)}{2}\sigma^2(z)z^{n-2} - R(z)z^n \\ &= z^{n-2}\left(nb(z)z + \frac{n(n-1)}{2}\sigma^2(z) - R(z)z^2\right) \in F_n \end{aligned}$$

and, since the term in brackets is a polynomial, we have

$$nb(z)z + \frac{n(n-1)}{2}\sigma^2(z) - R(z)z^2 \in F_2 \subseteq F_4. \tag{8}$$

Similarly, since $A_{n-1} \in F_n$ and $A_{n-2} \in F_n$ we have

$$(n-1)b(z)z + \frac{(n-1)(n-2)}{2}\sigma^2(z) - R(z)z^2 \in F_3 \subseteq F_4 \tag{9}$$

$$(n-2)b(z)z + \frac{(n-2)(n-3)}{2}\sigma^2(z) - R(z)z^2 \in F_4. \tag{10}$$

Since

$$\begin{aligned} \sigma^2(z) = &\left(nb(z)z + \frac{n(n-1)}{2}\sigma^2(z) - R(z)z^2\right) + \left((n-2)b(z)z + \frac{(n-2)(n-3)}{2}\sigma^2(z) - R(z)z^2\right) \\ &- 2\left((n-1)b(z)z + \frac{(n-1)(n-2)}{2}\sigma^2(z) - R(z)z^2\right) \end{aligned}$$

inclusions (8), (9) and (10) together yield

$$\sigma^2 \in F_4 \tag{11}$$

Similarly, since

$$\begin{aligned} zb(z) = &\left(nb(z)z + \frac{n(n-1)}{2}\sigma^2(z) - R(z)z^2\right) - \left((n-1)b(z)z + \frac{(n-1)(n-2)}{2}\sigma^2(z) - R(z)z^2\right) \\ &- (n-1)\sigma^2 \end{aligned}$$

inclusions (8), (9) and (11) together yield

$$b \in F_3. \tag{12}$$

Finally, inclusions (8), (11) and (12) together yield

$$R \in F_2.$$

Recall that A_n is of degree at most n. Now substituting $R(z) = \sum_{k=0}^{2} R_k z^k$, $b(z) = \sum_{k=0}^{3} b_k z^k$, $\sigma^2(z) = \sum_{k=0}^{4} a_k z^k$ into the definition of A_n, and setting the coefficient of z^{n+2} to zero yields

$$nb_3 + \frac{n(n-1)}{2}a_4 = R_2 \tag{13}$$

Similarly, equating to zero the coefficient of z^{n+1} in the expansion of A_n yields

$$nb_2 + \frac{n(n-1)}{2}a_3 = R_1.$$

Finally, equating to zero the coefficient of z^{n+1} in the expansion of A_{n-1} yields

$$(n-1)b_3 + \frac{(n-1)(n-2)}{2}a_4 = R_2 \tag{14}$$

Note that equations (13) and (14) together are equivalent to

$$R_2 = \frac{n}{2}b_3 = -\frac{n(n-1)}{2}a_4.$$

Finally, substituting these expressions into equation (7) and comparing the coefficients of the monomials z^k yields the system of ODEs for the functions for g_k. □ □

3. Two examples

In this section we further explore the properties of the polynomial term structure models via two examples. However, we begin this section with a simple observation. Consider a general factor term structure model as described in the introduction, such that $R(z) \geq 0$ for all $z \in I$ where $I \subseteq \mathbb{R}^d$ is the state space of the factor process Z. Let the discounted bond prices be defined by

$$\tilde{P}_t(T) = e^{-\int_0^t R(Z_s)ds} H(T-t, Z_t)$$

and suppose that $\tilde{P}(T)$ is a local martingale for all $T \geq 0$. Firstly, if $0 \leq H(x,z) \leq 1$ for all $x \geq 0, z \in I$, then $\tilde{P}(T)$ is a true martingale. Indeed, note that the right-hand side takes values in the bounded interval $[0,1]$, and recall that bounded local martingales are true martingales. Conversely, if $\tilde{P}(T)$ is a true martingale, then

$$H(x,z) = \mathbb{E}[e^{-\int_0^x R(Z_s)ds} | Z_0 = z]$$

and hence necessarily we have the bound $0 \leq H(x,z) \leq 1$ for all (x,z). From these considerations, we focus on term structure models where the function H is bounded.

Consider the case $d = 1$ and suppose that the function $H : \mathbb{R}_+ \times I \to \mathbb{R}$ takes the polynomial form of equation (4). If H is bounded, it must be the case that the state space $I \subseteq \mathbb{R}$ of the factor process is bounded. (However, notice that the boundedness of H does not imply the boundedness of I in the case of exponential polynomial models.)

With these preliminaries out of the way, we now consider our first example.

Example 1. Here the spot rate process solves the SDE

$$dr_t = \alpha(\beta - r_t)dt + \sqrt{r_t(k-r_t)(\ell - r_t)}dW_t$$

with parameter $\alpha > 0$ and $0 < \beta < k < \ell$. Roughly speaking, the dynamics of interest rate in this model resemble the Cox–Ingersoll–Ross process when r_t is very small. The parameters β intuitively plays the role of a long time mean level, while α controls the speed of mean reversion. However, in this model, the interest rate stays within the bounded interval $I = [0,k]$.

Indeed, Feller's test for explosions shows that

$$\mathbb{P}(0 < r_t < k \text{ for all } t \geq 0) = 1$$

as long as the initial condition r_0 is in $(0,k)$ and if

$$\frac{\alpha\beta}{kl} \geq \frac{1}{2} \text{ and } \frac{\alpha(k-\beta)}{k(\ell - k)} \geq \frac{1}{2}$$

Notice that this is a quadratic $n = 2$ model. The corresponding matrix S of this family takes the form

$$S = \begin{pmatrix} 0 & \alpha\beta & 0 \\ -1 & -\alpha & 2\alpha\beta + k\ell \\ 0 & -1 & -2\alpha - k - \ell \end{pmatrix}$$

from which the function $G = (g_0, g_1, g_2)^\top$ can be calculated by solving the ODE $\dot{G} = SG$ subject to $G(0) = (1,0,0)^\top$.

Notice that the process is ergodic in the pricing measure $\mathbb{Q}$, and its invariant density f is given by the unique stationary solution of the corresponding Fokker–Planck PDE

$$\begin{aligned} f(r) &= \frac{C}{\sigma(r)^2} e^{\int_{r_0}^r \frac{2b(\rho)}{\sigma(\rho)^2}d\rho} \\ &\propto r^{2\zeta - 1}(k-r)^{2\eta - 1}(\ell - r)^{-2\theta - 1} \end{aligned}$$

where

$$\zeta = \frac{\alpha\beta}{k\ell},\ \ \eta = \frac{\alpha(k-\beta)}{k(\ell-k)},\ \ \theta = \frac{\alpha(\ell-\beta)}{\ell(\ell-k)}$$

and where $C > 0$ is such that $\int_0^k f(r)dr = 1$.

The characteristic polynomial of matrix S is given by

$$f(\lambda) = \lambda^3 + (3\alpha+k+l)\lambda^2 + (\alpha(2\alpha+k+l)+3\alpha\beta+kl)\lambda + \alpha\beta(2\alpha+k+l)$$

Notice that

$$\begin{aligned}
f(0) &= \alpha\beta(2\alpha+k+l) && > 0\\
f(-\beta) &= \beta(k-\beta)(\beta-l) && < 0\\
f(-(\alpha+k)) &= (\alpha+k)(\alpha k-3\alpha\beta)+\alpha\beta(2\alpha+k+l)\\
&\geq (\alpha+k)(\alpha k-3\alpha\beta)+2\alpha\beta(\alpha+k)\\
&= \alpha(\alpha+k)(k-\beta) && > 0\\
f(-(2\alpha+k+l)) &= -2\alpha\beta-kl && < 0
\end{aligned}$$

Hence the equation $f(\lambda) = 0$ has three distinct negative roots. Therefore the matrix S will always has three negative eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that

$$-(2\alpha+k+l) < \lambda_3 < -(\alpha+k) < \lambda_2 < -\beta < \lambda_1 < 0$$

From a calibration to US Treasury rates from 2006 to 2014, the parameters $\beta = 0.03, \alpha = 0.5, k = 0.1, l = 0.2$ fit reasonably well. Notice that this choice of parameters satisfies Feller's condition, therefore the corresponding spot rate process will not hit the boundary. The corresponding matrix is given by

$$S = \begin{pmatrix} 0 & 0.015 & 0 \\ -1 & -0.5 & 0.05 \\ 0 & -1 & -1.3 \end{pmatrix}.$$

With the above parameter values, we can simulate the spot rate process r. A typical sample path with initial a very low spot rate $r_0 = 10^{-4}$ in line with current market conditions, is illustrated in Figure 1.

The initial yield curve, calculated by the formula

$$y_0(T) = -\frac{1}{T}\log H(T, r_0),$$

is given in Figure 2

By changing the initial spot rate r_0, we can also get different shapes of yield curve as shown in Figure 3.

The graphs of the functions g_k and the polynomials P_i are shown in Figures 4 and 5 where

$$H(x,r) = \sum_{k=0}^{2} g_k(x) r^k = \sum_{i=0}^{2} P_i(r) e^{\lambda_i x}$$

where $\lambda_i \in \{-0.0294, -0.5377, -1.2329\}$ are the eigenvalues of S.

The next example where the factor has the interpretation as the square-root of the spot rate. The dynamics are chosen in such a way that the eigenvalues of the S matrix can be computed explicitly.

Example 2. Now let Z solve the following SDE

$$dZ_t = (Z_t - k)(Z_t + 2k + \alpha)(Z_t - 2k - \alpha)dt + \sqrt{Z_t^3(2k - Z_t)}dW_t$$

and define the spot rate by

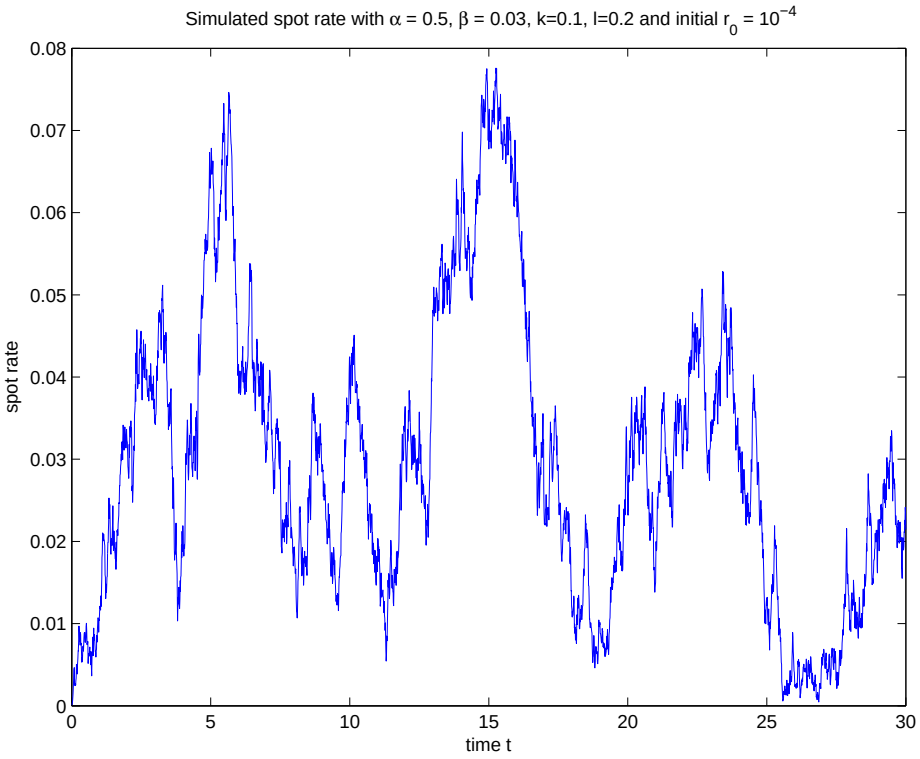
$$r_t = Z_t^2.$$

Feller's test for explosion says that the factor process Z_t will stay in the open interval $(0, 2k)$ as long as

$$\frac{\alpha(4k+\alpha)}{8k^2} \geq \frac{1}{2}.$$

The explicit eigenvalues of the matrix S are given by $-(2k+\alpha)^2$, $-(2k+\alpha)(2k+\alpha \pm \sqrt{\alpha^2+4k\alpha+2k^2})$.

FIGURE 1



In the following, we let $\alpha = 0.6, k = 0.2$, again calibrated from US Treasury rates. This choice of parameter satisfies Feller's condition, so spot rate process will not hit the boundary. The matrix S is given by

$$S = \begin{pmatrix} 0 & 0.2 & 0 \\ 0 & -1 & 0.4 \\ -1 & -0.2 & -2 \end{pmatrix}.$$

With the above parameter values, the simulated the spot rate process $(r_t)_{t\geq 0}$ and yield curve with $r_0 = 10^{-4}$ are shown in Figures 6 and 7:

By changing the initial spot rate r_0, we can also get different shapes of yield curve as shown in Figure 8. The corresponding eigenvalues are $-0.0408, -1.9592, -1$. The graph of the functions g_k and P_i are shown in Figures 9 and 10:

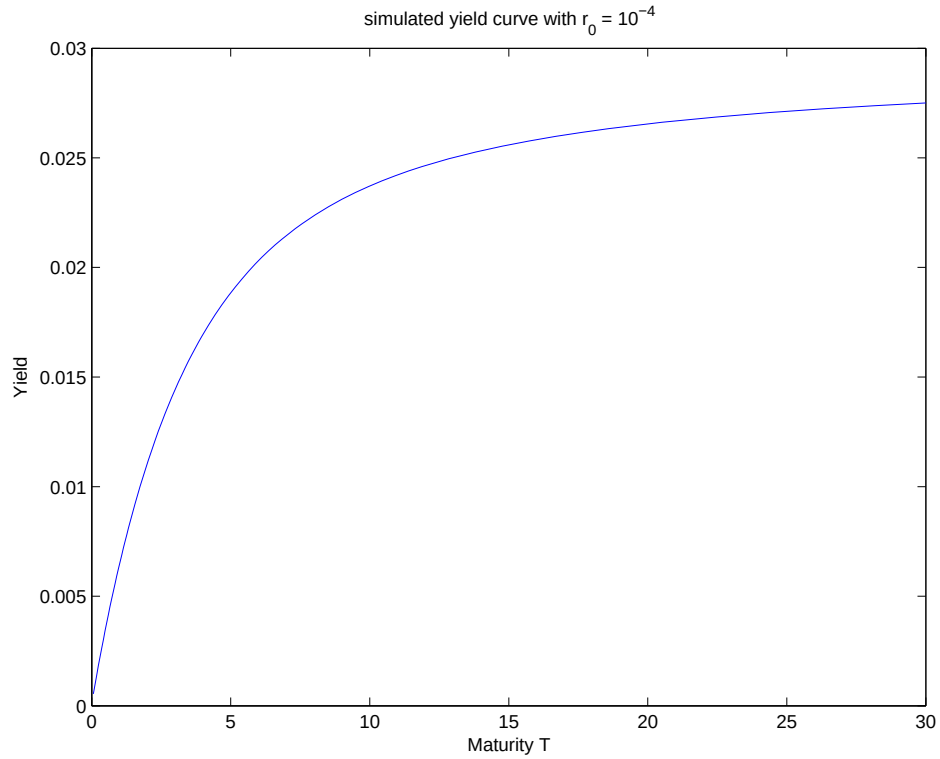
Notice that both examples above have real eigenvalues, hence the bond price can be viewed as a linear combination of bond prices with fixed interest rates given by the corresponding eigenvalues.

4. EXTENSIONS

In this section, we will extend theorem 2.1 in two different ways: namely allowing time dependency and allowing a multi-dimensional factor process.

4.1. **Hull-White extension.** As usual, by incorporating time-dependent parameters à la Hull–White, we can hope to have better model calibration. We introduce time dependency both in the dynamics of the factor process $(Z_t)_{t\geq 0}$ and the coefficient functions g_k. As one may expect, we will establish a similar sufficient and necessary condition in this case.

FIGURE 2



To be clear, we now consider a factor process $(Z_t)_{t\geq 0}$ be a non-explosive solution to the following time-inhomogeneous SDE

$$dZ_t = b(t,Z_t)dt + \sigma(t,Z_t)dW_t.$$

The spot rate r_t is modelled as $r_t = R(t,Z_t)$ and the bond price and discounted bond price are defined by

$$P_t(T) = \sum_{k=0}^{n} g_k(t,T)Z_t^k$$
$$\tilde{P}_t(T) = e^{-\int_0^t R(s,Z_s)ds}P_t(T)$$

where $g_k : \Delta \to \mathbb{R}$ are smooth deterministic functions satisfying the boundary conditions

$$g_0(T,T) = 1$$
$$g_k(T,T) = 0 \text{ for all } 1 \leq k \geq n,$$

where $\Delta = \{(t,T) : 0 \leq t \leq T\}$.

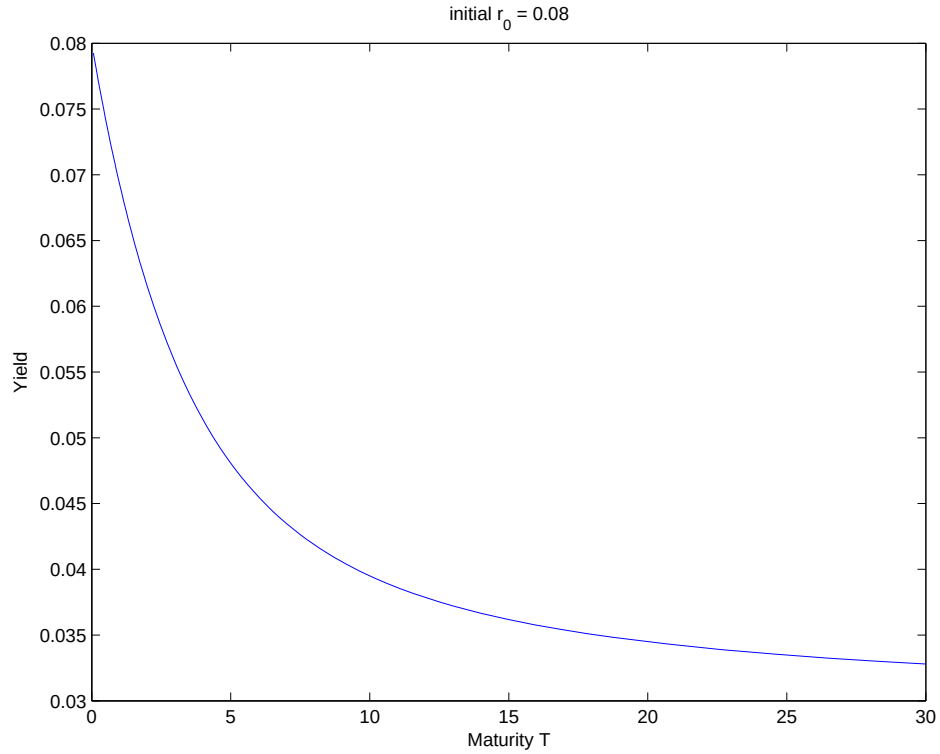
By adding the t component, the we are lead to the PDE

$$\sum_{i=0}^{n} \frac{\partial g_k}{\partial t}(t,T)z^k = \sum_{k=0}^{n} g_k(t,T)A_k(t,z) \quad \text{for all } (t,T,z) \tag{15}$$

where $A_k(t,z)$ are defined by

$$A_k(t,z) = R(t,z)z^k - kb(t,z)z^{k-1} - \frac{k(k-1)}{2}a(t,z)z^{k-2}$$
$$a(t,z) = \sigma^2(t,z).$$

FIGURE 3



Theorem 4.1. Suppose that $n \geq 2$ and that the functions $g_k(t,\cdot)$ are linearly independent for all $t \geq 0$. Then we must have $R(t,z) = \sum_{k=0}^{2} R_k(t)z^k$, $b(t,z) = \sum_{k=0}^{3} b_k(t)z^k$, $a(t,z) = \sum_{k=0}^{4} a_k(t)z^k$ where the coefficients satisfy

$$\begin{aligned}
\frac{n(n-1)}{2}a_4(t) + nb_3(t) - R_2(t) &= 0 \\
\frac{(n-1)(n-2)}{2}a_4(t) + (n-1)b_3(t) - R_2(t) &= 0 \\
\frac{n(n-1)}{2}a_3(t) + nb_2(t) - R_1(t) &= 0.
\end{aligned}$$

and the coefficient functions g_k are determined by the unique solutions to the ODE

$$\begin{aligned}
\frac{\partial}{\partial t}G(t,T) &= S(t)G(t,T) \\
G(T,T) &= (1,0,\ldots,0)^\top
\end{aligned}$$

and the $(n+1)\times(n+1)$ matrix $S(t)$ is defined by

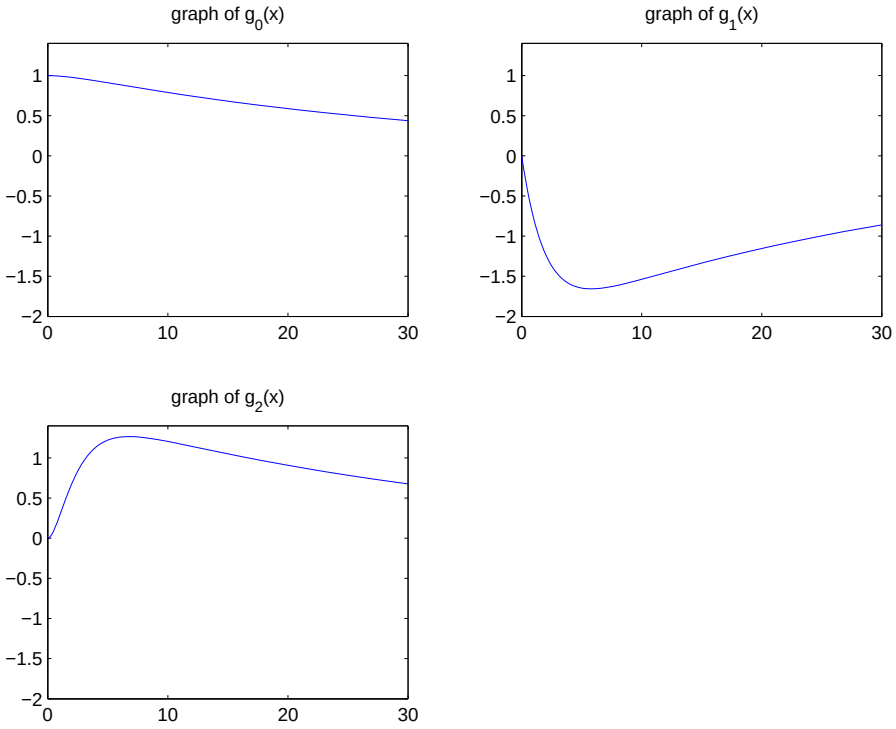
$$S_{j+k,j}(t) = R_k(t) - jb_{k+1}(t) - \frac{j(j-1)}{2}a_{k+2}(t).$$

Proof. Fix any $t \geq 0$, and choose $t < T_0 < \ldots < T_n$. The consistency condition (15) becomes $n+1$ linear equations in $n+1$ variables $A_k(t,z)$. Since the $g_k(t,\cdot)$ are linearly independent, there is a unique solution to this linear system, with the $A_k(t,z)$ written as a linear combination of the monomials $1,z,\ldots,z^n$ with t-dependent coefficients.

The rest of the proof goes exactly the same as the time-independent case. □

4.2. **Multi-dimensional factor process.** In this section, we will extend the polynomial model framework by allowing the factor process $(Z_t)_{t\geq 0}$ and the background Brownian motion $(W_t)_{t\geq 0}$ to be multi-dimensional. To be more specific,

FIGURE 4



let $(W_t)_{t\geq 0}$ be a D-dimensional Brownian motion and let $(Z_t)_{t\geq 0}$ be the factor process taking values in $I \subseteq \mathbb{R}^d$, assuming to be the (non-explosive) solution of the SDE

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dW_t$$

for some continuous deterministic functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times D}$. We define the diffusion function $a = \sigma\sigma^\top$, and note that the only role played by the parameter D is as the upper bound on the rank of the matrix $a(z)$.

For $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d_+$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$, we define the monomial z^k as follows:

$$z^k := z_1^{k_1} \cdots z_d^{k_d}$$

We define the total degree of k to be $|k| = k_1 + \ldots + k_d$, and set $K_n = \{k \in \mathbb{Z}^d_+ : |k| \leq n\}$. With the notation defined above, we let the bond price to be

$$P_t(T) = \sum_{k\in K_n} g_k(T-t)Z_t^k$$

where the functions g_k satisfy the boundary condition

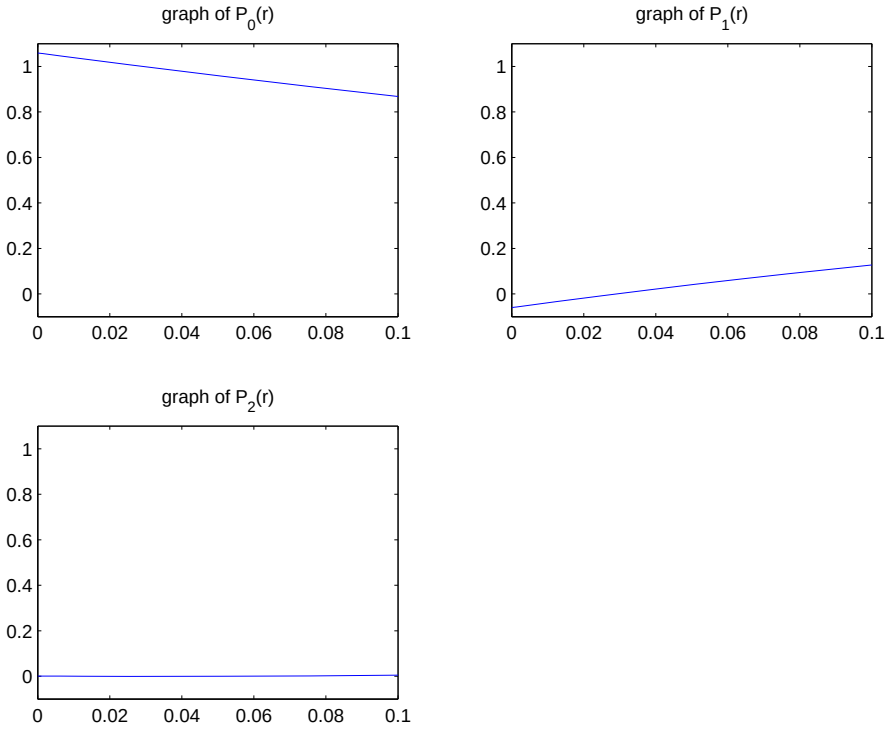
$$g_k(0) = 1 \quad \text{if } |k| = 0$$
$$g_k(0) = 0 \quad \text{otherwise}$$

The spot rate is modelled similarly as $r_t = R(Z_t)$ for some deterministic function $R : \mathbb{R}^d \to \mathbb{R}$.

The no-arbitrage condition turns out to be the condition:

$$\sum_{k\in K_n} \dot{g}_k(x)z^k = \sum_{k\in K_n} g_k(x)A_k(z) \tag{16}$$

FIGURE 5



holds for any $x \geq 0$ and $z \in I$, where the functions A_k are defined as

$$A_k(z) = \sum_{i=1}^{d} b_i(z) \frac{\partial(z^k)}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(z) \frac{\partial^2(z^k)}{\partial z_i \partial z_j} - R(z) z^k$$

Finally we define the notation

$$F_n = \left\{ f(z) : \sum_{k \in K_n} f_k z^k, \quad f_k \in \mathbb{R} \right\}$$

to be the family of polynomials in d variables of total degree less or equal to n.

Theorem 4.2. Suppose $n \geq 2$ and that the functions g_k are linearly independent. Then we must have $R \in F_2$, $b_i \in F_3 \quad 1 \leq i \leq d$ and $a_{ij} \in F_4 \quad 1 \leq i, j \leq d$. Furthermore, the coefficients are constrained in such a way that $A_k \in F_n$ for all k such that $|k| \in \{n-1, n\}$.

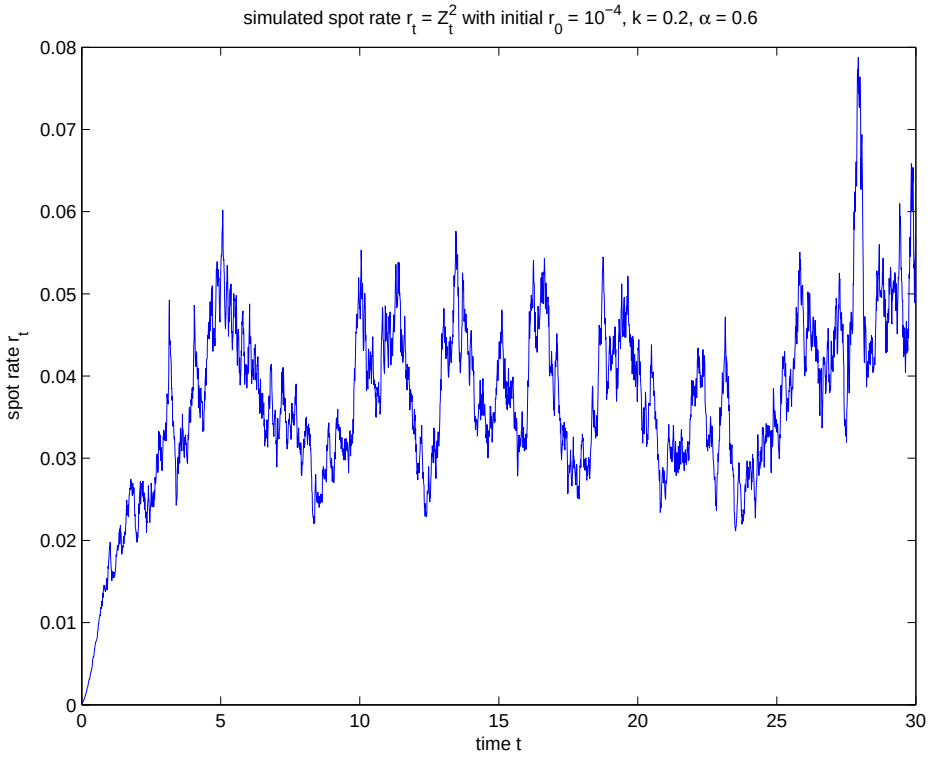
Proof. First we show that the functions $A_k \in F_n$ are polynomials for all $k \in K_n$. Let $N = |K_n|$ be the cardinality of the set K_n. Since the functions g_k are linearly independent, we can find N distinct points $x_1, \ldots, x_N$ independent of z such that the matrix with i-th column formed by the vector $(g_k(x_i), k \in K_n)$ is non-singular. Now fix any z, we can rewrite the no-arbitrage condition (16) as a set of N simultaneous linear equations in the N unknowns $A_k(z)$. Therefore the solution exists and is unique and can be written as linear combinations of the monomials z^k, hence all of the $A_k(z)$ are polynomials in d variables of total degree less or equal to n.

For ease of notation, we introduce the following definition:

$(a)_i = (0, \ldots, 0, a, 0, \ldots, 0)$ where a is the i-the component.

$(a,b)_{i,j} = (0, \ldots, a, \ldots, b, \ldots, 0)$ where a is the i-the component and b is the j-the component.

FIGURE 6



Since we must have $A_k \in F_n$ for all $k \in K_n$, we can conclude for any $1 \leq i, j \leq d$

$$\begin{aligned} A_0(z) &= -R(z) && \in F_n \\ A_{(1)_i}(z) &= b_i(z) - z_i R(z) && \in F_n \\ A_{(1,1)_{i,j}}(z) &= b_i(z) z_j + b_j(z) z_i + a_{ij}(z) - z_i z_j R(z) && \in F_n \end{aligned}$$

Therefore we may conclude immediately that the functions R, b_i, a_{ij} are polynomials. On the other hand

$$A_{(n)_i}(z) = n z_i^{n-1} b_i(z) + \frac{n(n-1)}{2} z_i^{n-2} a_{ii}(z) - z_i^n R(z) \in F_n$$

by cancelling the z_i^{n-2} factor, we may deduce that

$$n z_i b_i(z) + \frac{n(n-1)}{2} a_{ii}(z) - z_i^2 R(z) \in F_2 \tag{17}$$

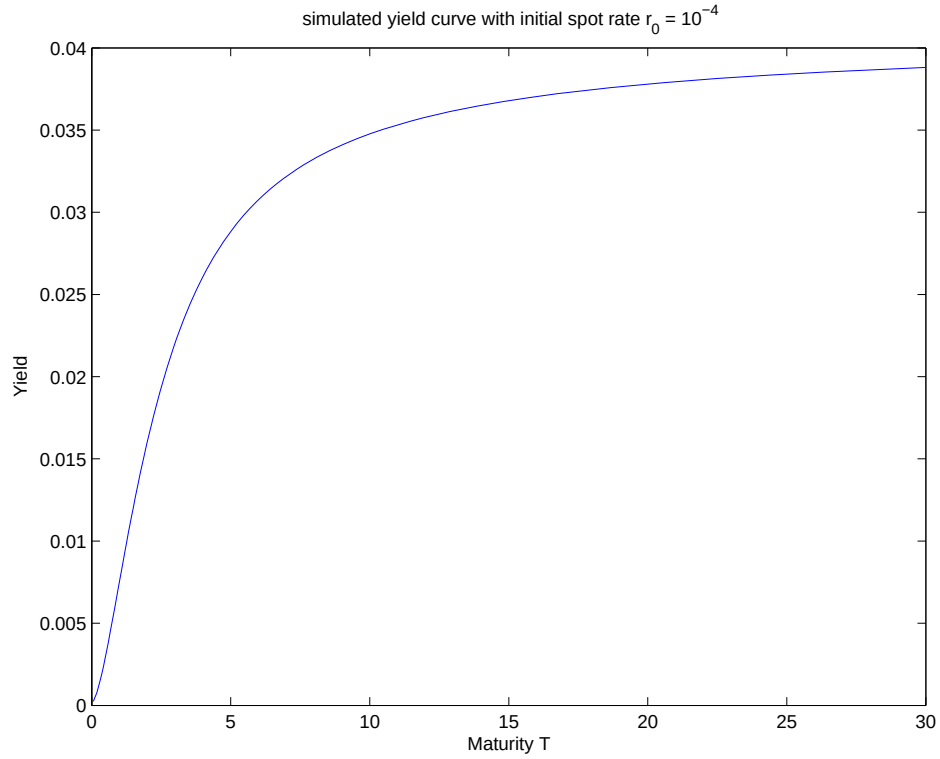
Similarly by considering $A_{(n-1)_i}, A_{(n-2)_i}, A_{(n-1,1)_{i,j}}$, we get

$$(n-1) z_i b_i(z) + \frac{(n-2)(n-1)}{2} a_{ii}(z) - z_i^2 R(z) \in F_3 \tag{18}$$

$$(n-2) z_i b_i(z) + \frac{(n-2)(n-3)}{2} a_{ii}(z) - z_i^2 R(z) \in F_4 \tag{19}$$

$$(n-1) z_i z_j b_i(z) + z_i^2 b_j(z) + \frac{(n-2)(n-1)}{2} z_j a_{ii}(z) + (n-1) z_i a_{ij}(z) - z_i^2 z_j R(z) \in F_3 \tag{20}$$

FIGURE 7



Subtracting (17) from (18) and subtracting (18) from (19) gives

$$z_i b_i(z) + (n-1)a_{ii}(z) \in F_3$$
$$z_i b_i(z) + (n-2)a_{ii}(z) \in F_4$$

Hence we get the required degree constraint on functions R, b, a. For the remaining part of this theorem, observe that given the degree constraint, the functions A_k will automatically in F_n as long as $|k| \leq n-2$. □

Remark 6. We note that the case when $n = 1$ essentially is covered in the paper of Siegel.[11]

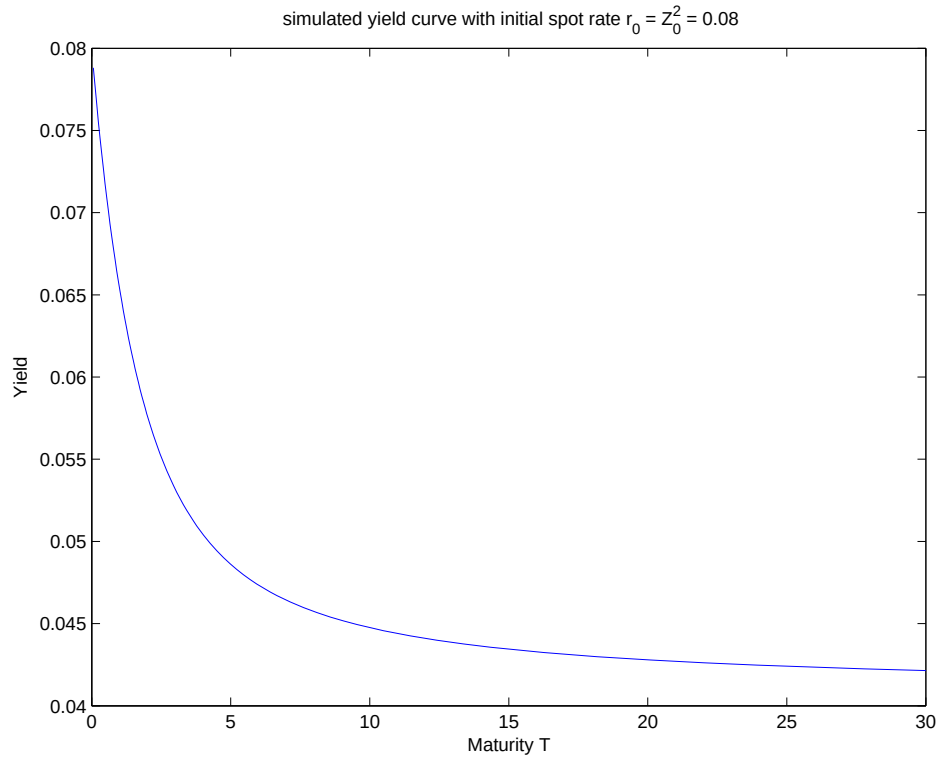
5. ACKNOWLEDGEMENT

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REFERENCES

[1] L. Chen, D. Filipovic and H.V. Poor. Quadratic term structure models for risk-free and defaultable rates. *Mathematical Finance* **14**(4): 515-536 (2004)
[2] J.C. Cox, J.E. Ingersoll and S.A. Ross. A theory of the term structure of interest rates. *Econometrica* **53**: 385-407 (1985)
[3] Ch. Cuchiero, M. Keller-Ressel, and J. Teichmann. *Polynomial processes and their applications to mathematical finance. Finance and Stochastics* **16**: 711-740 (2012)
[4] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *Annals of Applied Probability* **13**(3): 984-1053 (2003)
[5] D. Duffie and R. Kan. Multifactor models of the term structure. In *Mathematical Models in Finance*. Chapman & Hall (1995)
[6] D. Filipovic. Separable term structures and the maximal degree problem. *Mathematical Finance* **12**(4): 341-349 (2002)
[7] D. Filipovic and M. Larsson. Polynomial Preserving diffusions and applications in finance. *Pre-print* (2014)
[8] F. Jamshidian. Bond, futures and option evaluation in the quadratic interest rate model. *Applied Mathematical Finance* **3**: 93-115 (1996)

FIGURE 8



[9] M. Leippold and L. Wu. Asset pricing under the quadratic class. *Journal of Financial and Quantitative Analysis* **37**(2): 271-295 (2002)

[10] F.A. Longstaff. A nonlinear general equilibrium model of the term structure of interest rates. *Journal of Financial Economics* **23**: 195-224 (1989)

[11] A.F.Siegel. Price-admissibility conditions for arbitrage-free linear price function models for the term structure of interest rates. *Mathematical Finance* (2014)

[12] O. Vasicek. An equilibrium characterisation of the term structure. *Journal of Financial Economics* **5**(2): 177-188. (1977)

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FIGURE 9

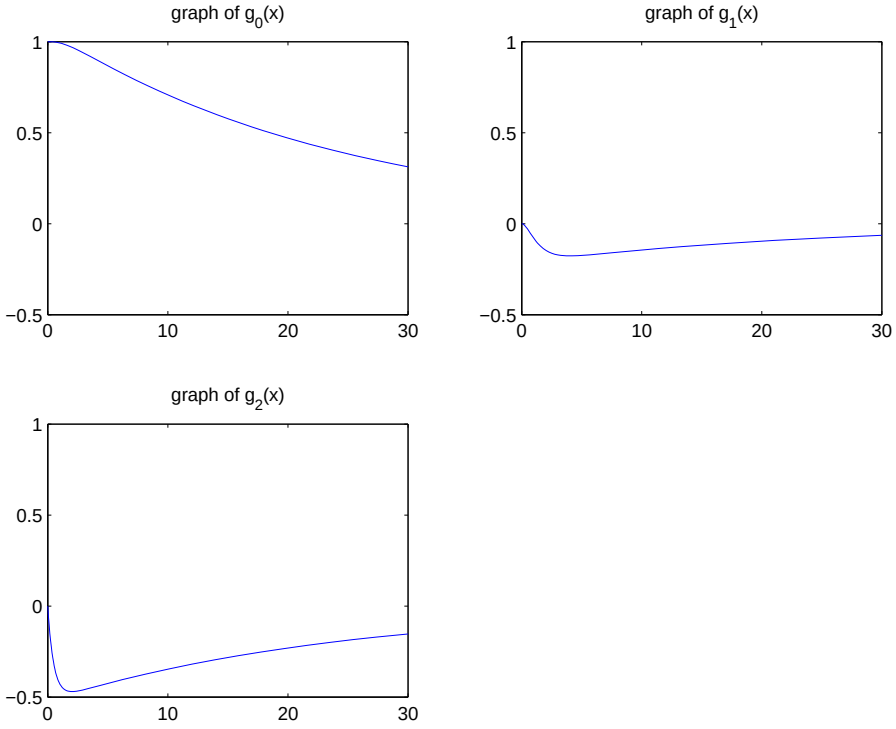


FIGURE 10

