OPTIMAL INVESTMENT: Examples 2

1 . Find the value function and optimal policy for the Merton consumption problem with $u(x) = \log(x)$.

2 . An investor may invest in a single log-Brownian asset, and in a riskless bank account bearing a constant interest rate r. His wealth evolves as

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt$$

where c_t is his consumption rate at time t, and θ_t is the cash value of his holding of the risky asset at time t. If his objective is to maximize $E\int_0^\infty e^{-\rho t}U(c_t)\,dt$, where $U(x) = -\gamma^{-1}\exp(-\gamma x)$, derive the HJB equation for his value function V. Show that the HJB equation is solved by $V(w) = -A\exp(-r\gamma w)$ for some constant $A > 0$. Show that this solution of the HJB equation implies that the agent's optimal consumption rate is of the form $c_t^* = rw_t + c_0^*$ for some constant c_0^*, and that the optimal investment strategy is to keep a fixed monetary value invested in the risky asset.

3 . So what problem did you *actually* solve in Question 1?

4 . An agent may invest in a single log-Brownian asset,

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

and in a riskless bank account bearing a constant interest rate r. Additionally, he receives a stream $(\varepsilon_t)_{t\geq 0}$ of endowment. Suppose that the agent's objective is to maximize $E\int_0^\infty e^{-\beta t}U(c_t)\,dt$, where $U'(x) = x^{-R}$ for some $R > 0$ different from 1. Solve his problem in each of the following two cases:

1. $\varepsilon_t = a$ constant for all $t \geq 0$;
2. the endowment evolves as

$$d\varepsilon_t = \varepsilon_t(vd\tilde{W}_t + \alpha dt)$$

for constants v and α, where $\tilde{W}$ is another Brownian motion, $dWd\tilde{W} = \rho dt$.

5 . An agent may invest in a riskless bank account yielding interest at rate r, or in a stock evolving as

$$dS_t = S_t(\sigma dW_t + \mu dt).$$

His objective is to maximise

$$E\Big[\int_0^\infty U(t, c_t)\, dt\Big],$$

where U is CRRA. If he chooses to follow a proportional strategy, where he invests a fixed proportion π of his wealth in the stock, and consumes at rate γw_t for some constant γ, find the value of his objective.

Suppose now that the true value of μ is unknown; the agent knows only that $\mu \in J$ for some interval J which contains r. If the agent wishes to maximise

$$\min_{\mu \in J} E\Big[\int_0^\infty U(t, c_t)\, dt\Big],$$

show that his optimal rule is to invest nothing in the stock, and that

$$\max_{c,\theta} \min_{\mu \in J} E\Big[\int_0^\infty U(t, c_t)\, dt\Big] = \min_{\mu \in J} \max_{c,\theta} E\Big[\int_0^\infty U(t, c_t)\, dt\Big]$$

6 . Suppose we have the usual dynamics, but now the objective of the agent is given as

$$\sup E \int_0^\infty e^{-\rho t} U(\xi_t)\, dt, \tag{0.1}$$

where the process ξ solves

$$d\xi_t = \lambda(c_t^\alpha - \xi_t)dt \tag{0.2}$$

for positive constants α and λ, and U is CRRA. Find the optimal policy of the agent as completely as you can.

7 . An agent works until a time τ of his choosing, at which time he retires. While working, he receives a constant income stream of ε, which incurs a disutility $\lambda > 0$. He invests his wealth in a riskless bank account bearing interest rate r, and in a risky stock with constant volatility σ and rate of growth μ. His wealth therefore evolves as

$$dw_t = rw_t dt + \theta(\sigma dW_t + (\mu - r)dt) - c_t dt + \varepsilon I_{\{t \le \tau\}} dt$$

(where W is a standard Brownian motion, and θ_t is the time-t value of his holding of the stock) and he seeks to maximise

$$E \int_0^\infty e^{-\rho t}(U(c_t) - \lambda I_{\{t \le \tau\}})\, dt.$$

Assume that $U'(x) = x^{-R}$ for some positive constant $R \neq 1$. Show that the critical level at which he retires is $\gamma_M^{-1}(\varepsilon/\lambda)^{1/R}$. By introducing the dual variable $z = V'(w)$, solve his problem as completely as you can.

8 . As a variant on the Constantinides habit formation example, let us suppose that the historical level of consumption is measured (as before) by

$$x_t = e^{-at}x_0 + b\int_0^t e^{a(s-t)}c_s\, ds,$$

but that the objective is to obtain

$$V(w,x) \equiv \sup E\Big[\int_0^\infty e^{-\rho t}u(c_t/x_t)dt\,\Big|\, w_0 = w, x_0 = x\Big].$$

Find the HJB equation for V. Can you solve it?

9 . Suppose that an agent lives for a random time τ which is independent of the evolution of the assets, and has a distribution specified in terms of its (deterministic) hazard rate $h : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$P[\tau > t] = \exp(-\int_0^t h(s)\, ds) \qquad (t > 0). \tag{0.3}$$

The agent's objective is to maximize

$$E\Big[\int_0^\tau \varphi(s)U(c_s)\, ds\Big] \tag{0.4}$$

where φ is some deterministic function which reflects the agent's preferences over the different times of consumption; for example, it may be that the agent cares more about consumption in his old age. What is the agent's optimal behaviour?

10 . Suppose that there is some Markov chain ξ taking values in the finite set $I = \{1, 2, \ldots, N\}$, which is independent of the driving Brownian motion W. We let Q denote the $N \times N$ matrix of jump intensities. The volatility and the growth rate of the stock depend on the value of ξ, so that the dynamics of the single risky asset become

$$dS_t/S_t = \sigma(\xi_t)dW_t + \mu(\xi_t)dt \tag{0.5}$$

for some functions σ, μ of the chain, and the wealth dynamics become

$$dw_t = rw_tdt + \theta_t\sigma(\xi_t)(dW_t + \kappa(\xi_t)dt) - c_tdt, \tag{0.6}$$

where $\kappa(\xi) = \sigma^{-1}(\mu(\xi) - r)$ is the market price of risk. Suppose that the values $\sigma(\xi), \xi \in I$ are distinct, and that the agent aims to maximise the usual objective

$$\sup E \int_0^\infty e^{-\rho t} u(c_t)\, dt, \tag{0.7}$$

where u is CRRA. Find the form of his optimal solution as explicitly as you can.

How would the problem (and its solution) change if $\sigma(\xi)$ was the *same* for all $\xi \in I$?