OPTIMAL INVESTMENT

The first five examples are basic background on Brownian motion, the remainder are stochastic optimal control questions.

1 . (i) If W is a Brownian motion, prove that for any $a \in \mathbb{R}$ the process $M_t = \exp(aW_t - \frac{1}{2}a^2 t)$ is a martingale, in two ways:

1. Use distributional properties of Brownian motion to calculate $E[M_t \,|\, \mathcal{F}_s\,]$ for $0 \leq s \leq t$;

2. Use Itô's formula to discover that M is a local martingale, and then establish estimates sufficient to prove that M is in fact a martingale.

(ii) Using (i) or otherwise, prove that $W_t^2 - t$ is a martingale.

(iii) If X is a continuous real-valued process such that for every $a \in \mathbb{R}$ the process $\exp(aX_t - \frac{1}{2}a^2 t)$ is a martingale, prove that X is a Brownian motion.

(iv) If we use the martingale $M_t = \exp(aW_t - \frac{1}{2}a^2 t)$ to define a new probability measure $\tilde{P}$ by the recipe

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = M_t,$$

prove that under $\tilde{P}$ the process $\tilde{W}_t = W_t - at$ is a Brownian motion.

2 . If X is a Brownian motion in $\mathbb{R}^n$, $n > 1$, started at 0, prove that

$$M_t \equiv |X_{1+t}| - \int_0^t \frac{n-1}{2|X_{1+s}|}\, ds$$

is a local martingale. What is its covariation process? In the case $n = 3$, show that

(i) $Y_t \equiv 1/|X_{1+t}|$ is a local martingale;

(ii) Y is bounded in L^2;

(iii) Y is not a martingale.

3 . If X, Y are independent Brownian motions, find the Itô expansions of the following semimartingales:

(i) Y_t/X_t;

(ii) $\tan^{-1}(Y_t/X_t)$;
(iii) $(X_t, Y_t)/\sqrt{X_t^2 + Y_t^2}$;
(iv) $X_t/(X_t^2 + Y_t^2)$.
In each case, comment on any possible issues concerning the application of Itô's formula.

4 . If $Z_t = X_t + iY_t$ is a complex Brownian motion (so X and Y are independent standard Brownian motions), and f is an analytic function, prove that $f(Z)$ is a local martingale. What is its covariation process? Find the Itô expansion of $\log Z_t$.

5 . Suppose that the real-valued process X solves the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0 \in \mathbb{R},$$

where $\sigma > 0$ and b are Lipschitz functions. Assuming that σ^{-2} is locally integrable, prove that $s(X_t)$ is a local martingale, where

$$s'(x) = \exp\left(-\int^x \frac{2b(y)}{\sigma(y)^2}\, dy \right)$$

and deduce that for $a < x_0 < b$, and $\tau = \inf\{t : X_t \notin (a, b)\}$

$$P(X(\tau) = b) = \frac{s(x_0) - s(a)}{s(b) - s(a)}.$$

6 . A real-valued process x evolves as

$$dx_t = \sigma dW_t + (u_t - \beta x_t)dt \qquad (1)$$

where the control process u is adapted and $\beta > 0$. If the objective is to obtain

$$\inf\ E\left[\int_0^\infty e^{-\rho t}\{\tfrac{1}{2}\lambda u_t^2 + \tfrac{1}{2}x_t^2\}\, dt \right]$$

where $\lambda > 0, \rho > 0$ are given constants, find the optimal solution as explicitly as you can.

7 . With the same dynamics (1), find the solution to

$$\sup E \exp\left(-\int_0^T \tfrac{1}{2}(ax_t^2 + bu_t^2)\, dt - \tfrac{1}{2}qx_T^2\right)$$

as explicitly as you can. [Here, a, b and q are all strictly positive.]

8 . Again with asset dynamics (1), suppose that now the objective is to minimize

$$E \int_0^\infty e^{-\rho t} \left\{ \tfrac{1}{2} a x_t^2 + \lambda |u_t| \right\} dt$$

where $a, \lambda > 0$. A natural conjecture is that the value function has the form

$$\begin{aligned} V(x) &= A + Bx^2 \qquad\qquad (|x| \leq x_*) \\ &= A + Bx_*^2 + \lambda(|x| - |x_*|) \qquad (|x| > x_*) \end{aligned}$$

for some constants A, B and x_* which would be chosen to make V continuously differentiable. Explain why this conjecture is false, and how you would go about finding the true value function.

9 . Take Example 8 and suppose now that u must be bounded: $|u_t| \leq 1$ for all t. What is the optimal solution now?

10 . With standard multivariate asset dynamics

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

and objective

$$E\left[\int_0^T U(t, c_t)\, dt + U(T, w_T) \right],$$

where

$$U(t,c) = a e^{-\rho t} u(c) \quad (0 \leq t < T), \quad U(T,c) = b u(c)$$

and $u(c) = c^{1-R}/(1-R)$ for some $R > 0$, $R \neq 1$, prove that

$$V(t,w) = f(t) u(w), \qquad \pi_t = \pi_M \, w_t, \qquad c_t = \gamma(t)\, w_t,$$

where

$$\begin{aligned} f(t) &= \left\{ b^{1/R} e^{-q(T-t)/R} + \frac{R a^{1/R}}{\rho + q} e^{-\rho t/R} \left(1 - e^{-(\rho+q)(T-t)/R} \right) \right\}^R \\ \pi_M &= R^{-1} (\sigma \sigma^T)^{-1} (\mu - r\mathbf{1}) \\ \gamma(t) &= a^{1/R} e^{-\rho t/R} f(t)^{-1/R} \end{aligned}$$

and

$$q \equiv (R-1)(r + |\kappa|^2/2R), \quad \kappa \equiv \sigma^{-1}(\mu - r\mathbf{1}).$$