OPTIMAL INVESTMENT: Examples 2

1 . An investor may invest in a single log-Brownian asset, and in a riskless bank account bearing a constant interest rate r. His wealth evolves as

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt$$

where c_t is his consumption rate at time t, and θ_t is the cash value of his holding of the risky asset at time t. If his objective is to maximize $E\int_0^\infty e^{-\rho t}U(c_t)\ dt$, where $U(x) = -\gamma^{-1}\exp(-\gamma x)$, derive the HJB equation for his value function V. Show that the HJB equation is solved by $V(w) = -A\exp(-r\gamma w)$ for some constant $A > 0$. Show that this solution of the HJB equation implies that the agent's optimal consumption rate is of the form $c_t^* = rw_t + c_0^*$ for some constant c_0^*, and that the optimal investment strategy is to keep a fixed monetary value invested in the risky asset.

2 . So what problem did you *actually* solve in Question 1?

3 . An agent may invest in a single log-Brownian asset,

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

and in a riskless bank account bearing a constant interest rate r. Additionally, he receives a stream $(\varepsilon_t)_{t\geq 0}$ of endowment. Suppose that the agent's objective is to maximize $E\int_0^\infty e^{-\beta t}U(c_t)\ dt$, where $U'(x) = x^{-R}$ for some $R > 0$ different from 1. Solve his problem in each of the following two cases:

1. $\varepsilon_t = a$ constant for all $t \geq 0$;
2. the endowment evolves as

$$d\varepsilon_t = \varepsilon_t(v d\tilde{W}_t + \alpha dt)$$

for constants v and α, where $\tilde{W}$ is another Brownian motion, $dW d\tilde{W} = \rho dt$.

4 . An agent may invest in a riskless bank account yielding interest at rate r, or in a stock evolving as

$$dS_t = S_t(\sigma dW_t + \mu dt).$$

His objective is to maximise

$$E\Big[\int_0^\infty U(t,c_t)\,dt\Big],$$

where U is CRRA. If he chooses to follow a proportional strategy, where he invests a fixed proportion π of his wealth in the stock, and consumes at rate γw_t for some constant γ, find the value of his objective.

Suppose now that the true value of μ is unknown; the agent knows only that $\mu \in J$ for some interval J which contains r. If the agent wishes to maximise

$$\min_{\mu\in J} E\Big[\int_0^\infty U(t,c_t)\,dt\Big],$$

show that his optimal rule is to invest nothing in the stock, and that

$$\max_{c,\theta}\min_{\mu\in J} E\Big[\int_0^\infty U(t,c_t)\,dt\Big] = \min_{\mu\in J}\max_{c,\theta} E\Big[\int_0^\infty U(t,c_t)\,dt\Big]$$

5 . Suppose we have the usual dynamics, but now the objective of the agent is given as

$$\sup E\int_0^\infty e^{-\rho t}U(\xi_t)\,dt, \tag{0.1}$$

where the process ξ solves

$$d\xi_t = \lambda(c_t^\alpha - \xi_t)dt \tag{0.2}$$

for positive constants α and λ, and U is CRRA. Find the optimal policy of the agent as completely as you can.

6 . An agent works until a time τ of his choosing, at which time he retires. While working, he receives a constant income stream of ε, which incurs a disutility $\lambda > 0$. He invests his wealth in a riskless bank account bearing interest rate r, and in a risky stock with constant volatility σ and rate of growth μ. His wealth therefore evolves as

$$dw_t = rw_t dt + \theta(\sigma dW_t + (\mu - r)dt) - c_t dt + \varepsilon I_{\{t\le\tau\}}dt$$

(where W is a standard Brownian motion, and θ_t is the time-t value of his holding of the stock) and he seeks to maximise

$$E\int_0^\infty e^{-\rho t}(U(c_t) - \lambda I_{\{t\le\tau\}})\,dt.$$

Assume that $U'(x) = x^{-R}$ for some positive constant $R \neq 1$. Show that the critical level at which he retires is $\gamma_M^{-1}(\varepsilon/\lambda)^{1/R}$. By introducing the dual variable $z = V'(w)$, solve his problem as completely as you can.