Theorem 3.1.2. *For all non-negative measurable functions f, g and all constants $\alpha, \beta \geq 0$,*

(a) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$,
(b) $f \leq g$ *implies* $\mu(f) \leq \mu(g)$,
(c) $f = 0$ *a.e. if and only if* $\mu(f) = 0$.

Proof. Define simple functions f_n, g_n by

$$f_n = (2^{-n}\lfloor 2^n f \rfloor) \wedge n, \quad g_n = (2^{-n}\lfloor 2^n g \rfloor) \wedge n.$$

Then $f_n \uparrow f$ and $g_n \uparrow g$, so $\alpha f_n + \beta g_n \uparrow \alpha f + \beta g$. Hence, by monotone convergence,

$$\mu(f_n) \uparrow \mu(f), \quad \mu(g_n) \uparrow \mu(g), \quad \mu(\alpha f_n + \beta g_n) \uparrow \mu(\alpha f + \beta g).$$

We know that $\mu(\alpha f_n + \beta g_n) = \alpha\mu(f_n) + \beta\mu(g_n)$, so we obtain (a) on letting $n \to \infty$. As we noted above, (b) is obvious from the definition of the integral. If $f = 0$ a.e., then $f_n = 0$ a.e., for all n, so $\mu(f_n) = 0$ and $\mu(f) = 0$. On the other hand, if $\mu(f) = 0$, then $\mu(f_n) = 0$ for all n, so $f_n = 0$ a.e. and $f = 0$ a.e.. $\square$

Theorem 3.1.3. *For all integrable functions f, g and all constants $\alpha, \beta \in \mathbb{R}$,*

(a) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$,
(b) $f \leq g$ *implies* $\mu(f) \leq \mu(g)$,
(c) $f = 0$ *a.e. implies* $\mu(f) = 0$.

Proof. We note that $\mu(-f) = -\mu(f)$. For $\alpha \geq 0$, we have

$$\mu(\alpha f) = \mu(\alpha f^+) - \mu(\alpha f^-) = \alpha\mu(f^+) - \alpha\mu(f^-) = \alpha\mu(f).$$

If $h = f + g$ then $h^+ + f^- + g^- = h^- + f^+ + g^+$, so

$$\mu(h^+) + \mu(f^-) + \mu(g^-) = \mu(h^-) + \mu(f^+) + \mu(g^+)$$

and so $\mu(h) = \mu(f) + \mu(g)$. That proves (a). If $f \leq g$ then $\mu(g) - \mu(f) = \mu(g - f) \geq 0$, by (a). Finally, if $f = 0$ a.e., then $f^\pm = 0$ a.e., so $\mu(f^\pm) = 0$ and so $\mu(f) = 0$. $\square$

Note that in Theorem 3.1.3(c) we lose the reverse implication. The following result is sometimes useful:

Proposition 3.1.4. *Let $\mathcal{A}$ be a π-system containing E and generating $\mathcal{E}$. Then, for any integrable function f,*

$$\mu(f 1_A) = 0 \text{ for all } A \in \mathcal{A} \quad \text{implies} \quad f = 0 \text{ a.e..}$$

Here are some minor variants on the monotone convergence theorem.

Proposition 3.1.5. *Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions, with $f_n \geq 0$ a.e.. Then*

$$f_n \uparrow f \text{ a.e.} \quad \Longrightarrow \quad \mu(f_n) \uparrow \mu(f).$$

Thus the pointwise hypotheses of non-negativity and monotone convergence can be relaxed to hold almost everywhere.

Proposition 3.1.6. *Let $(g_n : n \in \mathbb{N})$ be a sequence of non-negative measurable functions. Then*

$$\sum_{n=1}^{\infty} \mu(g_n) = \mu\left(\sum_{n=1}^{\infty} g_n\right).$$

This reformulation of monotone convergence makes it clear that it is the counterpart for the integration of functions of the countable additivity property of the measure on sets.

3.2. **Integrals and limits.** In the monotone convergence theorem, the hypothesis that the given sequence of functions is non-decreasing is essential. In this section we obtain some results on the integrals of limits of functions without such a hypothesis.

Lemma 3.2.1 (Fatou's lemma)**.** *Let $(f_n : n \in \mathbb{N})$ be a sequence of non-negative measurable functions. Then*

$$\mu(\liminf f_n) \leq \liminf \mu(f_n).$$

Proof. For $k \geq n$, we have

$$\inf_{m \geq n} f_m \leq f_k$$

so

$$\mu(\inf_{m \geq n} f_m) \leq \inf_{k \geq n} \mu(f_k) \leq \liminf \mu(f_n).$$

But, as $n \to \infty$,

$$\inf_{m \geq n} f_m \uparrow \sup_n \left(\inf_{m \geq n} f_m\right) = \liminf f_n$$

so, by monotone convergence,

$$\mu(\inf_{m \geq n} f_m) \uparrow \mu(\liminf f_n).$$

□

Theorem 3.2.2 (Dominated convergence)**.** *Let $(f_n : n \in \mathbb{N})$ be a sequence of integrable functions with $f_n \to f$ pointwise as $n \to \infty$. Suppose that, for some integrable function g*

$$|f_n| \leq g, \quad \textit{for all } n.$$

Then f is integrable and $\mu(f_n) \to \mu(f)$ as $n \to \infty$.

Proof. The limit f is measurable and $|f| \leq g$, so $\mu(|f|) \leq \mu(g) < \infty$, so f is integrable. We have $0 \leq g \pm f_n \to g \pm f$ so certainly $\liminf(g \pm f_n) = g \pm f$. By Fatou's lemma,

$$\mu(g) + \mu(f) = \mu(\liminf(g + f_n)) \leq \liminf \mu(g + f_n) = \mu(g) + \liminf \mu(f_n),$$
$$\mu(g) - \mu(f) = \mu(\liminf(g - f_n)) \leq \liminf \mu(g - f_n) = \mu(g) - \limsup \mu(f_n).$$

Since $\mu(g) < \infty$, we can deduce that

$$\mu(f) \leq \liminf \mu(f_n) \leq \limsup \mu(f_n) \leq \mu(f).$$

This proves that $\mu(f_n) \to \mu(f)$ as $n \to \infty$. □