Probability and Measure, Hand-out 2
Non-measurable subsets of $\mathbb{R}$

The *Axiom of Choice* states the following. Let S be a set, and let $\mathcal{A} = \{A_\alpha : \alpha \in \mathcal{I}\}$ be a collection of disjoint non-empty subsets of S. There exists a subset R of S containing exactly one element of each A_α, $\alpha \in \mathcal{I}$.

Let $I = [0, 1)$, and let $\mathcal{L}$ be the σ-field of Lebesgue-measurable subsets of I. We shall show that there exists a subset R of I such that $R \notin \mathcal{L}$.

Define the equivalence relation $\sim$ on I by $x \sim y$ if $x - y$ is rational. Let $\mathcal{A} = \{A_\alpha : \alpha \in \mathcal{I}\}$ be the equivalence classes of this equivalence relation. By the Axiom of Choice, we may find a set of representatives, say $R = \{c_\alpha : \alpha \in \mathcal{I}\}$ where $c_\alpha \in A_\alpha$ for each α. We call c_α the *representative* of A_α. Assume R is measurable (in that $R \in \mathcal{L}$). Now,

$$I = \bigcup_{q \in \mathbb{Q} \cap [0,1)} (q + R) \tag{1}$$

where $\mathbb{Q}$ is the rationals and addition is done modulo 1. We note two facts about the sets $q + R$, $q \in \mathbb{Q} \cap [0, 1)$.

1. They are disjoint. If $x \in (q + R) \cap (q' + R)$, then $x = q + c_1 = q' + c_2 \pmod 1$ for representatives c_1, c_2. Therefore $c_1 - c_2 = q' - q \pmod 1$ is rational, whence c_1, c_2 lie in the same equivalence class. But R contains exactly one representative of each class, so $c_1 = c_2$ and therefore $q = q' \pmod 1$.

2. The $q+R$ are measurable (since each is a translation, modulo 1, of the measurable R). They have the same measure, as follows. Let $q, q' \in \mathbb{Q} \cap [0, 1)$ satisfy $q < q'$. Define the mapping $f : I \to I$ by $f(x) = x + q' - q$ modulo 1. Thus f is a translation modulo 1. When restricted to the domain $q + R$, f is a one–one correspondence between $q + R$ and $q' + R$. Therefore, $\lambda(q + R) = \lambda(q' + R)$.

It follows that (1) is a countably-infinite disjoint union of sets with the same measure. By the countable additivity of Lebesgue measure λ, and the fact that $\lambda(I) = 1$, we can have neither $\lambda(R) = 0$ nor $\lambda(R) > 0$. This contradicts the assumption that $R \in \mathcal{L}$.