## Probability and Measure 3

**Note.** Unless otherwise specified, $(\Omega, \mathcal{F}, \mu)$ denotes a measure space, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

**1.** Let $g : \mathbb{R} \to \mathbb{R}$ be convex in the sense that:

$$g\big(tx + (1-t)y\big) \le tg(x) + (1-t)g(y), \qquad x, y \in \mathbb{R},\ 0 \le t \le 1.$$

Show that $g$ has supporting tangents in the sense that: for $x \in \mathbb{R}$, there exists $\lambda \in \mathbb{R}$ such that $g(y) \ge g(x) + \lambda(y - x)$ for all $y$.

**2.** Prove that the space $L^\infty(\Omega, \mathcal{F}, \mu)$ is complete.

**3.** Let $p \ge 1$, and let $f_n$, $f$ be measurable functions. Show that $f_n$ converges to $f$ in measure whenever $f_n \to f$ in $L^p$, but that the converse is not true.

**4.** Let $X$ be a random variable, $p \ge 1$, and write $Y = |X|$. Show that

$$\mathbb{E}(Y^p) = \int_0^\infty p x^{p-1} \mathbb{P}(Y \ge x)\, dx.$$

(a) If $\mathbb{E}(Y^p) < \infty$, show that $\mathbb{P}(Y \ge x) = \mathrm{o}(x^{-p})$ as $x \to \infty$.
(b) If $\mathbb{P}(Y \ge x) = \mathrm{o}(x^{-p})$, show that $\mathbb{E}(Y^q) < \infty$ for $1 \le q < p$.

**5.** Show that random variables $X$ and $Y$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are independent if and only if, for all measurable $g, h : \mathbb{R} \to \mathbb{R}$ such that $g(X)$, $h(Y)$ are integrable,

$$\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y)).$$

**6.** Let $X$ be a random variable whose moment generating function $M(t) = \mathbb{E}(e^{tX})$ is finite on some neighbourhood of the origin. Show that

$$\mathbb{P}(X > x) \le \inf_{t \in \mathbb{R}} \big\{ e^{-tx} M(t) \big\}, \qquad x > \mathbb{E}(X).$$

**7.** **(Continuation)** Let $X_1, X_2, \dots$ be independent identically distributed random variables with moment generating function $M(t)$ and mean value $m$. Assume that $M$ is finite on some neighbourhood of the origin. Show that $S_n = \sum_{i=1}^n X_i$ satisfies

$$\mathbb{P}(S_n > na) \le \Lambda(a)^n, \qquad a > m,$$

where $\Lambda(a) = \inf_{t \in \mathbb{R}} \big\{ e^{-ta} M(t) \big\}$. Compute $\Lambda(a)$ when the $X_i$ have the exponential distribution with parameter 1, and $a > 1$.

**8.** Let $(X_n : n \ge 1)$ be a sequence of identically distributed random variables in $L^2(\mathbb{P})$. Let $\epsilon > 0$. Show that $\mathbb{P}(|X_1| \ge \epsilon\sqrt{n}) = \mathrm{o}(n^{-1})$ as $n \to \infty$, and that

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} |X_k| \to 0 \qquad \text{in probability } \textbf{and} \text{ in } L^1.$$

**9.** Let $\mathcal{H}$ be a closed linear space of $L^2(\Omega, \mathcal{F}, \mu)$ with the $L^2$ norm $\|\cdot\| = \|\cdot\|_2$ and the inner product $\langle \cdot, \cdot \rangle$. Show the 'Pythagoras' and 'parallelogram rules',

$$\|f+g\|^2 = \|f\|^2 + 2\langle f, g\rangle + \|g\|^2,$$
$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2),$$

valid for $g, f \in \mathcal{H}$.

**10.** Let $V_1, V_2, \dots$ be an increasing sequence of closed subspaces of $L^2(\Omega, \mathcal{F}, \mu)$ such that $V_i \subseteq V_{i+1}$ for all $i$. For $f \in L^2$, write $f_n$ for the orthogonal projection of $f$ onto $V_n$. Show that $f_n$ converges in $L^2$.

**11.** Let $V$, $W$ be closed subspaces of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $V \subseteq W$. Show that $\mathbb{E}(\mathbb{E}(X \mid W) \mid V) = \mathbb{E}(X \mid V)$ for $X \in L^2$.

**12.** Find a uniformly integrable sequence $(X_n : n \geq 1)$ such that $X_n \to 0$ a.s. and $\mathbb{E}(\sup_n |X_n|) = \infty$.

**13.** Let $g : [0, \infty) \to [0, \infty)$ be an increasing function such that $g(x)/x \to \infty$ as $x \to \infty$. Show that $(X_n : n \geq 1)$ is uniformly integrable if $\sup_n \mathbb{E}(g(|X_n|)) < \infty$.

**14.** Show that the sum $X_n + Y_n$ of two uniformly integrable sequences forms a uniformly integrable sequence.

**15.** (a) Suppose that $X_n \to X$ in $L^r$ where $r \geq 1$. Show that $(|X_n|^r : n \geq 1)$ is uniformly integrable, and deduce that $\mathbb{E}(X_n^r) \to \mathbb{E}(X^r)$ if $r$ is an integer.

(b) Conversely, suppose that $(|X_n|^r : n \geq 1)$ is uniformly integrable where $r \geq 1$, and show that $X_n \to X$ in $L^r$ whenever $X_n \to X$ in probability.