Probability and Measure I

1. Let Ω be a set.
(a) Let $\mathcal{F}_i$, $i \in I$, be σ-fields of subsets of Ω. Show that $\mathcal{F} = \bigcap_i \mathcal{F}_i$ is a σ-field also. Your should not assume that I is countable.
(b) Give an example of σ-fields $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F} \cup \mathcal{G}$ is not a σ-field.

2. Show that a countably additive non-negative set function on a ring is both increasing and countably subadditive.

3. Let $\mathcal{F}$ be the set of all finite subsets of Ω, and let $\mathcal{G}$ be the set of all cofinite subsets (a subset is *cofinite* if and only if its complement is finite). Show that $\mathcal{F} \cup \mathcal{G}$ is a σ-field if and only if Ω is finite.

4. Let $\mathcal{F}$ be a class of subsets of Ω.
(a) Suppose that $\Omega \in \mathcal{F}$, and that $A \setminus B = A \cap B^{\mathrm{c}} \in \mathcal{F}$ for all $A, B \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.
(b) Suppose that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under the operations of complementation and taking finite disjoint unions. Show that $\mathcal{F}$ need not be a field.

5. Let $\mathcal{F}$ be a π-system and a λ-system. Show that $\mathcal{F}$ is a σ-field.

6. Show that the following families of subsets of $\mathbb{R}$ generate the same σ-field $\mathcal{B}$:
(i) $\{(a,b) : a < b\}$, (ii) $\{(a,b] : a < b\}$, (iii) $\{(-\infty, b] : b \in \mathbb{R}\}$.

7. A σ-field is called *separable* if it can be generated by a countable family of sets. Show that the Borel σ-field $\mathcal{B}$ of $\mathbb{R}$ is separable.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n : n \geq 1)$ be a sequence of sets in $\mathcal{F}$, and define

$$\liminf A_n = \bigcup_n \bigcap_{m \geq n} A_m, \quad \limsup A_n = \bigcap_n \bigcup_{m \geq n} A_m.$$

Show that $\mu(\liminf A_n) \leq \liminf_{n\to\infty} \mu(A_n)$. Show that

$$\mu(\limsup A_n) \geq \limsup_{n\to\infty} \mu(A_n) \qquad \text{if } \mu(\Omega) < \infty.$$

Give an example with $\mu(\Omega) = \infty$ when this inequality fails.

9. **(Completion)** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A subset N of Ω is called *null* if $N \subseteq B$ for some $B \in \mathcal{F}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of all null sets. Prove that the family of subsets

$$\mathcal{C} = \{A \cup N : A \in \mathcal{F},\ N \in \mathcal{N}\}$$

is a σ-field. Show that the measure μ may be extended to a measure μ' on $\mathcal{C}$. The σ-field $\mathcal{C}$ is called the *completion* of $\mathcal{F}$ with respect to μ.

The completion of the Borel σ-field of $\mathbb{R}$ with respect to Borel measure λ is called the *Lebesgue* σ-field of $\mathbb{R}$, and the corresponding extension λ' is called *Lebesgue measure*.

10. Let B be a Borel subset of $\mathbb{R}$ and let λ be Lebesgue measure. Suppose $\lambda(B) < \infty$. Show that, for every $\epsilon > 0$, there exists a finite union A of intervals such that $\lambda(A \triangle B) < \epsilon$.

11. **(Cantor set)** Let $C_0 = [0,1]$, and let $C_1, C_2, \ldots$ be constructed iteratively by deletion of middle-thirds. Thus,

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1], \quad C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1],$$

and so on. The set $C = \lim_{n\to\infty} C_n = \bigcap_n C_n$ is called the *Cantor set*. Let F_n be the distribution function of a random variable uniformly distributed on C_n.

(i) Show that C is uncountable and has Lebesgue measure 0.

(ii) Show that the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists for all $x \in [0,1]$.

(iii) Show that F is continuous on $[0,1]$ with $F(0) = 0$, $F(1) = 1$.

(iv) Show that F is differentiable except on a set of measure 0, and that $F'(x) = 0$ wherever F is differentiable.

12. Let $(A_n : n \geq 1)$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $A_n \in \mathcal{F}$ for all n. Show that the A_n, $n \geq 1$, are independent if and only if the σ-fields which they generate, $\mathcal{F}_n = \{\varnothing, A_n, A_n^c, \Omega\}$, are independent.

13. The Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s > 1.$$

Let $s > 1$ and let X, Y be independent random variables with $\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s)$. For $p \in \{1, 2, \ldots\}$ let A_p be the event that p divides X. Show that the events $\{A_p : p \text{ prime}\}$ are independent. Deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

Show that

$$\mathbb{P}(X \text{ is square-free}) = \frac{1}{\zeta(2s)},$$

$$\mathbb{P}(X,\, Y \text{ have highest common factor } n) = \frac{n^{-2s}}{\zeta(2s)}.$$

14. Let $X_1, X_2, \ldots$ be independent $N(0,1)$ random variables. Prove that

$$\limsup_n \frac{X_n}{\sqrt{2 \log n}} = 1 \quad \text{with probability 1.}$$

15. Let $X_1, X_2, \ldots$ be independent random variables with the uniform distribution on $[0,1]$. Let A_n be the event that a record occurs at time n, i.e., that $X_n > X_m$ for all $m < n$. Show that infinitely many records occur with probability 1.