1. Extension and Uniqueness

1.1. **Extension.** A set $\mathcal{R}$ of subsets of Ω is a *ring* if $\emptyset \in \mathcal{R}$, and $B \setminus A, A \cup B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$.

Note that rings are closed under intersection, since

$$A \cap B = C \setminus \{(C \setminus A) \cup (C \setminus B)\}, \quad C = A \cup B.$$

Let $\mathcal{R}$ be a ring. The set function $\mu : \mathcal{R} \to \mathbb{R}$ is called *countably subadditive* if

$$\mu\left(\bigcup_i A_i\right) \le \sum_i \mu(A_i) \quad \text{whenever} \quad A_1, A_2, \dots \in \mathcal{R}, \ \bigcup_i A_i \in \mathcal{R}.$$

Lemma 1.1.1. *The increasing set function μ is countably subadditive on $\mathcal{R}$ if and only if*

$$\mu(A) \le \sum_i \mu(A_i) \quad \textit{whenever} \quad A, A_1, A_2, \dots \in \mathcal{R}, \ A \subseteq \bigcup_i A_i.$$

Proof. That the condition of the lemma implies countable subadditivity is trivial, on taking $A = \bigcup_i A_i$. Assume then that μ is countably subadditive on $\mathcal{R}$. Let $A, A_1, A_2, \dots \in \mathcal{R}$ be such that $A \subseteq \bigcup_i A_i$. Then

$$A = \bigcup_i (A_i \cap A) \in \mathcal{R}.$$

Since $\mathcal{R}$ is a ring, $A_i \cap A \in \mathcal{R}$ for all i. By subadditivity and monotonicity,

$$\mu(A) = \mu\left(\bigcup_i (A_i \cap A)\right) \le \sum_i \mu(A_i \cap A) \le \sum_i \mu(A_i)$$

as required. $\square$

Theorem 1.1.2 (Carathéodory extension theorem)**.** *Let $\mathcal{R}$ be a ring of subsets of Ω and let $\mu : \mathcal{R} \to [0, \infty]$ be a countably additive set function. Then μ extends to a measure μ' on the σ-field $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.*

Proof of Carathéodory extension theorem. For any $B \subseteq \Omega$, define the *outer measure*

$$\mu^*(B) = \inf \sum_n \mu(A_n)$$

where the infimum is taken over all sequences $(A_n : n \in \mathbb{N})$ in $\mathcal{R}$ such that $B \subseteq \cup_n A_n$ and is taken to be ∞ if there is no such sequence. Note that μ^* is increasing and $\mu^*(\emptyset) = 0$. Let us say that $A \subseteq \Omega$ is *μ^*-measurable* if, for all $B \subseteq \Omega$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Write $\mathcal{M}$ for the set of all μ^*-measurable sets. *We shall show that* $\mathcal{M}$ *is a σ-field containing* $\mathcal{R}$ *and that* μ^* *is a measure on* $\mathcal{M}$*, extending* μ*.* This will prove the theorem.

Step I. We show that μ^* *is countably subadditive.* Suppose that $B \subseteq \cup_n B_n$. If $\mu^*(B_n) < \infty$ for all n, then, given $\varepsilon > 0$, there exist sequences $(A_{nm} : m \in \mathbb{N})$ in $\mathcal{R}$, with

$$B_n \subseteq \cup_m A_{nm}, \quad \mu^*(B_n) + \varepsilon/2^n \geq \sum_m \mu(A_{nm}).$$

Then

$$B \subseteq \cup_n \cup_m A_{nm}$$

so

$$\mu^*(B) \leq \sum_n \sum_m \mu(A_{nm}) \leq \sum_n \mu^*(B_n) + \varepsilon.$$

Hence, in any case,

$$\mu^*(B) \leq \sum_n \mu^*(B_n).$$

Step II. We show that μ^* *extends* μ*.* Since $\mathcal{R}$ is a ring and μ is countably additive, μ is countably subadditive. By Lemma 1.1.1, for $A \in \mathcal{R}$ and any sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{R}$ with $A \subseteq \cup_n A_n$,

$$\mu(A) \leq \sum_n \mu(A_n).$$

On taking the infimum over all such sequences, we see that $\mu(A) \leq \mu^*(A)$. On the other hand, it is obvious that $\mu^*(A) \leq \mu(A)$ for $A \in \mathcal{R}$.
Step III. We show that $\mathcal{M}$ *contains* $\mathcal{R}$*.* Let $A \in \mathcal{R}$ and $B \subseteq \Omega$. We have to show that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

By subadditivity of μ^*, it is enough to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

If $\mu^*(B) = \infty$, this is trivial, so let us assume that $\mu^*(B) < \infty$. Then, given $\varepsilon > 0$, we can find a sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{R}$ such that

$$B \subseteq \cup_n A_n, \quad \mu^*(B) + \varepsilon \geq \sum_n \mu(A_n).$$

Then

$$B \cap A \subseteq \cup_n (A_n \cap A), \quad B \cap A^c \subseteq \cup_n (A_n \cap A^c)$$

so

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n \mu(A_n \cap A^c) = \sum_n \mu(A_n) \leq \mu^*(B) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.
Step IV. We show that $\mathcal{M}$ is an field. Clearly $\Omega \in \mathcal{M}$ and $A^c \in \mathcal{M}$ whenever $A \in \mathcal{M}$. Suppose that $A_1, A_2 \in \mathcal{M}$ and $B \subseteq \Omega$. Then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\
&= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c).
\end{aligned}$$

Hence $A_1 \cap A_2 \in \mathcal{M}$.
Step V. We show that $\mathcal{M}$ is a σ-field and that μ^* is a measure on $\mathcal{M}$. We already know that $\mathcal{M}$ is an field, so it suffices to show that, for any sequence of disjoint sets $(A_n : n \in \mathbb{N})$ in $\mathcal{M}$, for $A = \cup_n A_n$ we have

$$A \in \mathcal{M}, \quad \mu^*(A) = \sum_n \mu^*(A_n).$$

So, take any $B \subseteq \Omega$, then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1) + \mu^*(B \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&= \cdots = \sum_{i=1}^{n} \mu^*(B \cap A_i) + \mu^*(B \cap A_1^c \cap \cdots \cap A_n^c).
\end{aligned}$$

Note that $\mu^*(B \cap A_1^c \cap \cdots \cap A_n^c) \geq \mu^*(B \cap A^c)$ for all n. Hence, on letting $n \to \infty$ and using countable subadditivity, we get

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap A^c) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

The reverse inequality holds by subadditivity, so we have equality. Hence $A \in \mathcal{M}$ and, setting $B = A$, we get

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

1.2. **Uniqueness.** A set $\mathcal{P}$ of subsets of Ω is called a *π-system* if $\emptyset \in \mathcal{P}$, and $A \cap B \in \mathcal{P}$ whenever $A, B \in \mathcal{P}$.

Theorem 1.2.1 (Uniqueness of extension)**.** *Let $\mathcal{P}$ be a π-system of subsets of Ω and let $\mathcal{E}$ be the σ-field generated by $\mathcal{P}$. Suppose that*

$$\mu_1 : \mathcal{E} \to [0, \infty], \quad \mu_2 : \mathcal{E} \to [0, \infty]$$

are measures on $\mathcal{E}$ with $\mu_1(\Omega) = \mu_2(\Omega) < \infty$. If $\mu_1 = \mu_2$ on $\mathcal{P}$, then $\mu_1 = \mu_2$ on $\mathcal{E}$.

In the proof, we shall make use of the following bit of nonsense. A set $\mathcal{L}$ of subsets of Ω is called a *d-system* if

(i) $\Omega \in \mathcal{L}$,
(ii) $B \setminus A \in \mathcal{L}$ whenever $A, B \in \mathcal{L}$ and $A \subseteq B$,
(iii) if $A_1, A_2, \dots \in \mathcal{L}$ satisfy $A_n \subseteq A_{n+1}$ for all n, then $\bigcup_i A_i \in \mathcal{L}$.

Lemma 1.2.2 (Dynkin π/d-system lemma)**.** *Let $\mathcal{R}$ be a π-system. Then any d-system containing $\mathcal{R}$ contains also the σ-field generated by $\mathcal{R}$.*

Proof of the uniqueness theorem. Consider $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$. By hypothesis, $\Omega \in \mathcal{D}$; for $A, B \in \mathcal{E}$ with $A \subseteq B$, we have

$$\mu_1(A) + \mu_1(B \setminus A) = \mu_1(B) < \infty, \quad \mu_2(A) + \mu_2(B \setminus A) = \mu_2(B) < \infty$$

so, if $A, B \in \mathcal{D}$, then also $B \setminus A \in \mathcal{D}$; if $A_n \in \mathcal{D}, n \in \mathbb{N}$, with $A_n \uparrow A$, then

$$\mu_1(A) = \lim_n \mu_1(A_n) = \lim_n \mu_2(A_n) = \mu_2(A)$$

so $A \in \mathcal{D}$. Thus $\mathcal{D}$ is a d-system containing the π-system $\mathcal{R}$, so $\mathcal{D} = \mathcal{E}$ by Dynkin's lemma. □

Proof of Dynkin π/d-system lemma. Denote by $\mathcal{D}$ the intersection of all d-systems containing $\mathcal{R}$. Then $\mathcal{D}$ is itself a d-system (easy). We shall show that $\mathcal{D}$ is also a π-system and hence a σ-field, thus proving the lemma. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{R}\}.$$

Then $\mathcal{R} \subseteq \mathcal{D}'$ because $\mathcal{R}$ is a π-system. Let us check that $\mathcal{D}'$ is a d-system: clearly $\Omega \in \mathcal{D}'$; next, suppose $B_1, B_2 \in \mathcal{D}'$ with $B_1 \subseteq B_2$, then for $A \in \mathcal{R}$ we have

$$(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$$

because $\mathcal{D}$ is a d-system, so $B_2 \setminus B_1 \in \mathcal{D}'$; finally, if $B_n \in \mathcal{D}', n \in \mathbb{N}$, and $B_n \uparrow B$, then for $A \in \mathcal{R}$ we have

$$B_n \cap A \uparrow B \cap A$$

so $B \cap A \in \mathcal{D}$ and $B \in \mathcal{D}'$. Hence $\mathcal{D} = \mathcal{D}'$.

Now consider

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{D}\}.$$

Then $\mathcal{R} \subseteq \mathcal{D}''$ because $\mathcal{D} = \mathcal{D}'$. We can check that $\mathcal{D}''$ is a d-system, just as we did for $\mathcal{D}'$. Hence $\mathcal{D}'' = \mathcal{D}$ which shows that $\mathcal{D}$ is a π-system as promised. □