g.r.grimmett@statslab.cam.ac.uk
http://www.statslab.cam.ac.uk/~grg/

MARKOV CHAINS

Example Sheet 4

4.1 Two fleas are bound together to take part in a nine-legged race on the vertices A, B, C of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with with exponential distribution, mean $1/\lambda$. Flea 2 behaves similarly, but hops in the anti-clockwise direction, the times between his hops having mean $1/\mu$. Show that the probability that they are at A at a given time $t > 0$ (starting from A at time $t = 0$) is

$$\frac{1}{3} + \frac{2}{3}\exp\left\{-\frac{3(\lambda+\mu)t}{2}\right\}\cos\left\{\frac{\sqrt{3}(\lambda-\mu)t}{2}\right\}.$$

4.2 Let $(X_t)_{t\geq 0}$ be a birth-death process with rates $\lambda_n = n\lambda$ and $\mu_n = n\mu$, and suppose $X_0 = 1$. Show that $h(t) = \mathbb{P}(X_t = 0)$ satisfies

$$h(t) = \int_0^t e^{-(\lambda+\mu)s}\{\mu + \lambda h(t-s)^2\}ds$$

and deduce that if $\lambda \neq \mu$ then

$$h(t) = \frac{\mu e^{\mu t} - \mu e^{\lambda t}}{\mu e^{\mu t} - \lambda e^{\lambda t}}.$$

4.3 Consider an immigration-death process X, being a birth-death process with birth rates $\lambda_n = \lambda$ and death rates $\mu_n = n\mu$; let Y be the jump chain. Find the transition matrix of Y, and show that it has as stationary distribution

$$\pi_n = \frac{1}{2(n!)}\left(1 + \frac{n}{\rho}\right)\rho^n e^{-\rho}$$

where $\rho = \lambda/\mu$. Explain why this differs from the stationary distribution of X.

4.4 Let $(X_t)_{t\geq 0}$ be an irreducible Markov chain with generator G. Show that the transition probabilities of $(X_t)_{t\geq 0}$ and its jump chain are related by

$$\int_0^\infty p_{i,i}(t)dt = \frac{1}{g_i}\sum_{n=0}^{\infty}\pi_{i,i}(n)$$

and deduce that $(X_t)_{t\geq 0}$ is transient if and only if, for some state i,

$$\int_0^\infty p_{i,i}(t)dt < \infty.$$

4.5 Customers arrive at a certain queue in a Poisson process with rate λ. The single 'server' has two states A and B, state A signifying that he is 'in attendance' and state B that he is having a tea-break. Independently of how many customers are in the queue, he fluctuates between these states as a Markov chain Y on $\{A, B\}$ with generator

$$\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The total service time of any customer is exponentially distributed with parameter μ and is independent of the chain Y and of the service times of other customers.

Describe the system as a Markov chain X with state-space

$$\{A_0, A_1, A_2, \dots\} \cup \{B_0, B_1, B_2, \dots\},$$

A_n signifying that the server is in state A and there are n people in the queue (including anyone being served) and B_n signifying that the server is in state B and there are n people in the queue.

Give a very brief explanation why, for some θ in $(0, 1]$, and $k = 0, 1, 2, \dots$,

$$P(X \text{ hits } A_0 \mid X_0 = A_k) = \theta^k.$$

Show that $(\theta - 1)f(\theta) = 0$, where

$$f(\theta) = \lambda^2\theta^2 - \lambda(\lambda + \mu + \alpha + \beta)\theta + (\lambda + \beta)\mu.$$

By considering $f(1)$ or otherwise, prove that X is transient if $\mu\beta < \lambda(\alpha + \beta)$, and explain why this is intuitively obvious.

4.6 Consider the continuous-time birth-death chain (X_t) with non-zero birth rates λ_i, $i \geq 0$, and death rates μ_i, $i \geq 1$. Suppose

$$\frac{\mu_i}{\lambda_i} = \left(\frac{i}{i+1}\right)^\alpha, \quad \mu_i = i^\beta, \quad i \geq 1,$$

with $\alpha, \beta \in \mathbb{R}$ and $\lambda_0 = 1$. Determine for which values of α and β the chain is transient, for which values it is null recurrent, and for which positive recurrent.

4.7 In each of the following cases, compute $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1)$ for the Markov chain $(X_t)_{t\geq 0}$ with the given generator on $\{1, 2, 3, 4\}$:

(a) $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix}$, (d) $\begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

4.8 Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process.

Write down the generator G of a continuous-time Markov chain which models this, explaining very briefly what the states of the chain represent. Calculate the essentially unique invariant measure for G, and deduce that the chain is positive recurrent if and only if $\lambda/\mu < \frac{1}{2}$.

4.9* Calls arrive at a telephone exchange as a Poisson process of rate λ, and the lengths of calls are independent exponentially-distributed random variables with parameter μ. Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process.

Show that for large t the distribution of the number of lines in use at time t is approximately Poisson with mean λ/μ.

Find the mean length of the busy periods during which at least one line is in use.

Show that the expected number of lines in use at time t, given that n are in use at time 0, is $ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$.

Show that, in equilibrium, the number N_t of calls finishing in the time interval $[0, t]$ has Poisson distribution of mean λt.

Is $(N_t)_{t\geq 0}$ a Poisson process?

Paper IV type questions: IIA only

Note that it is not necessarily the case that the Paper IV question will be a straight essay; as with other courses, there may be a slightly harder and/or longer problem to solve, or the question may mix an essay component and a problem component. The following give examples of the sort of essays that might arise. You should discuss essay writing with your supervisor — there is more to it than just regurgitating bits of your notes.

IV.1 Write an essay on the relation between discrete-time and continuous-time Markov chains. [*The mark scheme for this question indicates that candidates could have obtained marks by mentioning: Behaviour of the jump chain; Properties inherited from the jump chain; Derivation of backward equation from the jump chain; Invariant measures, positive recurrence; Discrete chain obtained by regular sampling of a continuous chain, convergence to equilibrium.*]

IV.2 Write an essay on transience and recurrence in discrete-time Markov chains.

IV.3 Write an essay on the Poisson process, including a discussion of the equivalence of the holding times, infinitesimal and transition probability definitions.

IV.4 Write an essay on the long-time behaviour of discrete-time Markov chains.