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MARKOV CHAINS

Example Sheet 3

3.1 Suppose that S and T are independent exponentially-distributed random variables with parameters α and β respectively. What is the distribution of $\min\{S, T\}$? What is the probability that $S \leq T$? Show that the two events $\{S < T\}$ and $\{\min\{S, T\} \geq t\}$ are independent.

3.2 Let $T_1, T_2, \ldots$ be independent exponentially-distributed random variables with parameter λ and let N be an independent geometric random variable with

$$\mathbb{P}(N = n) = \beta(1-\beta)^{n-1}, \qquad n = 1, 2, \ldots .$$

Show that $T = \sum_{i=1}^{N} T_i$ has an exponential distribution with parameter $\lambda\beta$.

3.3 Superposition. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities λ and μ. Show that the arrivals of flying objects form a Poisson process with intensity $\lambda + \mu$.

3.4 Thinning. Insects land in the soup in the manner of a Poisson process with intensity λ, and each such insect is green with probability p, independently of the colours of all other insects. Show that the arrivals of green insects form a Poisson process with intensity λp.

3.5 Let T_n be the time of the nth arrival in a Poisson process N with intensity λ, and define the excess lifetime process $E(t) = T_{N(t)+1} - t$, being the time one must wait subsequent to t before the next arrival. Show by conditioning on T_1 that

$$\mathbb{P}\big(E(t) > x\big) = e^{-\lambda(t+x)} + \int_0^t \mathbb{P}\big(E(t-u) > x\big)\lambda e^{-\lambda u}\, du.$$

Solve this integral equation in order to find the distribution function of $E(t)$. Explain your conclusion. [If necessary, you may assume that there is a unique eligible solution to the integral equation, but it is worth trying to do the problem without making this assumption.]

3.6 Let B be a process of simple birth with immigration with parameters λ and ν, and with $B(0) = 0$; the corresponding birth rates are $\lambda_n = n\lambda + \nu$. Write down the sequence of differential-difference equations for $p_n(t) = \mathbb{P}(B(t) = n)$. Without solving these equations, use them to show that $m(t) = \mathbb{E}B(t)$ satisfies $m'(t) = \lambda m(t) + \nu$, and solve for $m(t)$.

3.7 Arrivals of the Number 1 bus form a Poisson process with rate 1 bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate 7 buses per hour.
(a) What is the probability that exactly 3 buses pass by in 1 hour?
(b) What is the probability that exactly 3 Number 7 buses pass by while I am waiting for a Number 1?
(c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What then is the probability that I wait for 30 minutes without seeing a single bus?

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3.8 A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the number of vehicles that have passed by time t constitutes a Poisson process with rate λ, and suppose it takes a fixed time a to walk across the lane. Assuming the pedestrian can foresee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely? [*Hint*: Consider the time at which the first car passes.]

How long on average does it take to cross two similar lanes

(a) when one must walk straight across (assuming that the pedestrian will not cross if at any time whilst crossing a car passes in either direction), and

(b) when an island in the middle of the road makes it safe to stop half-way?

3.9 Each bacterium in a colony splits into two identical bacteria after a time having the exponential distribution with parameter λ; these bacteria then split in the same way but independently. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$. Show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$ satisfies

$$\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s}\phi(t-s)^2 ds.$$

Change variables $u = t - s$ in the integral and deduce that $d\phi/dt = \lambda\phi(\phi - 1)$.

[*Hint*: You may need to use the fact, which you should check, that $\frac{d}{dt}\int_0^t f(s,t)ds = f(t,t) + \int_0^t \frac{\partial}{\partial t} f(s,t)ds$.]

Hence deduce that, for $q = 1 - e^{-\lambda t}$, and $n = 1, 2, \ldots,$

$$\mathbb{P}(X_t = n) = q^{n-1}(1-q).$$

3.10 Let X be a Markov chain on $\{1, 2\}$ with generator

$$G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

where $\lambda\mu > 0$.

(a) Calculate G^n and hence find $\sum_{n=0}^{\infty}(t^n/n!)G^n$.

(b) Solve the equation $\boldsymbol{\pi} G = \mathbf{0}$ in order to find the stationary distribution. Verify that $p_{ij}(t) \to \pi_j$ as $t \to \infty$.

3.11* Let $(X_t)_{t\geq 0}$ be a Markov chain on the integers with transition rates $q_{i,i+1} = \lambda q_i$, $q_{i,i-1} = \mu q_i$, and $q_{i,j} = 0$ if $|j - i| \geq 2$, where $\lambda + \mu = 1$ and $\lambda, \mu, q_i > 0$ for all i. Find, for all i,

(a) the probability, starting from 0, that X_t ever hits i;

(b) the expected total time spent in state i, starting from 0.

In the case $\mu = 0$, write down a necessary and sufficient condition for $(X_t)_{t\geq 0}$ to be explosive. Why is this condition necessary for $(X_t)_{t\geq 0}$ to be explosive for all $\mu \in [0, \frac{1}{2})$?

Show that, in general, $(X_t)_{t\geq 0}$ is non-explosive if and only if one of the following conditions holds:

(i) $\lambda = \mu$,

(ii) $\lambda > \mu$ and $\sum_{i=1}^{\infty} 1/q_i = \infty$,

(iii) $\lambda < \mu$ and $\sum_{i=-\infty}^{-1} 1/q_i = \infty$.