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# MARKOV CHAINS

## Example Sheet 2

**2.1** Let $X$ be a Markov chain containing an absorbing state $s$ to which all other states lead (i.e., $j \to s$ for all $j$). Show that all states other than $s$ are transient.

**2.2** Let $X$ be a Markov chain on $\{0, 1, 2, \dots\}$ with transition matrix given by $p_{0,j} = a_j$ for $j \geq 0$, $p_{i,i} = r$ and $p_{i,i-1} = 1 - r$ for $i \geq 1$. Assume that $0 < r < 1$. Classify the states of the chain, and find their mean recurrence times. [You may find it useful to define $J = \sup\{j : a_j > 0\}$.]

**2.3** Compute $p_{11}(n)$ and classify the states of the Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}.$$

**2.4** A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability $\frac{1}{4}$, and moves to each of its neighbouring vertices with probability $\frac{1}{4}$. Let $v$ and $w$ be two diametrically opposite vertices. If the walk starts at $v$, find
(a) the mean number of steps until its first return to $v$,
(b) the mean number of steps until its first visit to $w$,
(c) the mean number of visits to $w$ before its first return to $v$.

**2.5** In Exercise 1.10, which states are recurrent and which are transient?

**2.6** What can be said about the number of visits to each state in the case where (a) a Markov chain is transient, and (b) a Markov chain is recurrent?

Consider the Markov chain $(X_n)_{n\geq 0}$ of Exercise 1.13. Show for this chain that $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty) = \mathbb{P}(\forall m,\ \exists n.\ X_N \geq m \text{ for all } N \geq n) = 1$.

Suppose the transition probabilities satisfy instead

$$p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}.$$

For each $\alpha \in (0, \infty)$ find the value of $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

**2.7** The rooted binary tree is an infinite graph $T$ with one distinguished vertex $R$ from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on $T$ jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

**2.8** Show (by projection onto $\mathbb{Z}^3$ or otherwise) that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

**2.9** Find all invariant distribution of the transition matrix in Exercise 1.10.

**2.10** Two containers A and B are placed adjacently to one another, and gas is allowed to pass through a small aperture joining them. There are $N$ molecules in all, and we assume that, at each epoch of time, one molecule (chosen at random) passes through the aperture. Show that the number of molecules in A evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain? [This is the 'Ehrenfest urn model', first introduced by Ehrenfest under the name 'dog-flea model'.]

**2.11** A fair die is thrown repeatedly. Let $X_n$ denote the sum of the first $n$ throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13)$$

quoting carefully any general theorems that you use.

**2.12** Find the invariant distributions of the transition matrices in Exercise 1.9, parts (a), (b) and (c), and compare them with your answers to that exercise.

**2.13** Each morning a student takes one of the three books (labelled 1, 2, 3) he owns from his shelf. The probability that he chooses the book with label $i$ is $\alpha_i$ (where $0 < \alpha_i < 1$, $i = 1, 2, 3$), and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If $p_n$ denotes the probability that on day $n$ the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, $p_n$ converges as $n \to \infty$, and determine the limit.

**2.14** In each of the following cases determine whether the stochastic matrix $P$ corresponds to a chain which is reversible in equilibrium:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$; (b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$;

(c) $I = \{0, 1, 2, \dots\}$ and $p_{01} = 1$, $p_{i,i+1} = p$, $p_{i,i-1} = 1 - p$ for $i \geq 1$.
(d) $p_{ij} = p_{ji}$ for all $i, j \in S$.

**2.15** A random walk on the set $\{0, 1, 2, \dots, b\}$ has transition matrix given by $p_{00} = 1 - \lambda_0$, $p_{bb} = 1 - \mu_b$, $p_{i,i+1} = \lambda_i$ and $p_{i+1,i} = \mu_{i+1}$ for $0 \leq i < b$, where $0 < \lambda_i, \mu_i < 1$ for all $i$, and $\lambda_i + \mu_i = 1$ for $1 \leq i < b$. Show that this process is time-reversible in equilibrium.

**2.16** Let $X$ be an irreducible non-null persistent aperiodic Markov chain. Show that $X$ is time-reversible in equilibrium if and only if

$$p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_2 j_1}$$

for all $n$ and all finite sequences $j_1, j_2, \dots, j_n$ of states.