Interacting particle systems II

1. Let $X_n, Y_n \in L^2(\Omega, \mathcal{F}, P)$ be such that $X_n \to X$, $Y_n \to Y$ in L^2. Show that $X_n Y_n \to XY$ in L^1. [Reminder: L^p is the set of random variables Z with $E(|Z|^p) < \infty$, and $Z_n \to Z$ in L^p if $E(|Z_n - Z|^p) \to 0$. You may use any standard fact such as the Cauchy–Schwarz inequality.]
2. (continuation) Consider bond percolation on the square lattice $\mathbb{Z}^2$. Let X and Y be increasing functions on the sample space, such that $E(X^2), E(Y^2) < \infty$. Show that X and Y are positively associated. [You make assume positive association for product measure on graphs with finitely many edges.]
3. Let P_p be a product measure on the space $\{0,1\}^n$ with density p; that is, P_p is the measure which governs the behaviour of n independent identically distributed random variables with distribution given by $P_p(Z = 1) = 1 - P_p(Z = 0) = p$. Show by induction on n that P_p satisfies the Harris–FKG inequality, which is to say that $P_p(A \cap B) \geq P_p(A) P_p(B)$ for any pair A, B of increasing events.
4. **Bond/site comparison**. Let G be an infinite connected graph with maximal vertex degree Δ. Show that the critical probabilities for bond and site percolation on G satisfy

   $$p_{\mathrm{c}}(\text{bond}) \leq p_{\mathrm{c}}(\text{site}) \leq 1 - \big(1 - p_{\mathrm{c}}(\text{bond})\big)^{\Delta}.$$

   [The first inequality is a fair target. The second is a little harder.]
5. Let a_n be the number of self-avoiding walks on $\mathbb{L}^d$, i.e., the number of paths $x_0, e_0, x_1, e_1, \dots, x_n$ of distinct vertices x_j and edges $e_j = \langle x_j, x_{j+1} \rangle$. Show that $a_{m+n} \leq a_m a_n$, and deduce the existence of the connective constant $\mu = \lim_{n\to\infty} \{a_n^{1/n}\}$. Show that $p_{\mathrm{c}}(\mathbb{Z}^d) \geq \mu^{-1}$. Use duality and the FKG inequality to obtain $p_{\mathrm{c}}(\mathbb{Z}^2) \leq 1 - \mu^{-1}$ when $d = 2$.
6. **Russo's formula**. Let X be a random variable on the finite sample space $\Omega = \{0,1\}^E$. Show that

   $$\frac{d}{dp} E_p(X) = \sum_{e \in E} E_p(\delta_e X)$$

   where $\delta_e X(\omega) = X(\omega^e) - X(\omega_e)$, and ω^e (respectively, ω_e) is the configuration obtained from ω by replacing $\omega(e)$ by 1 (respectively, 0).
   Let A be an increasing event, with indicator function I_A. An edge e is called *pivotal* for the event A in the configuration ω if $\delta_e I_A(\omega) = 1$. Show that the derivative of $P_p(A)$ equals the mean number of pivotal edges for A.
7. Let $\chi(p)$ be the mean number of vertices in the open cluster at the origin of bond percolation at density p, and suppose that p is such that $\chi(p) < \infty$. Show the existence of $\gamma > 0$ such that $P_p(0 \leftrightarrow \partial\Lambda_k) \leq e^{-\gamma k}$ for all k. [You are not required to re-prove the inequality

   $$g_n \leq \sum_{x \in \partial\Lambda_m} P_p(0 \leftrightarrow x) g_{n-m}$$

   proved already in lectures, more to fill in the details.]

8. **One-dimensional percolation**. Each edge of the one-dimensional lattice $\mathbb{L}$ is *open* with probability p, as usual. For $k \in \mathbb{Z}$, let $r(k) = \max\{u : k \leftrightarrow k+u\}$, and $R_n = \max\{r(k) : 1 \le k \le n\}$. Show that $P(R_n > u) \le np^u$, and hence, for $\epsilon > 0$,

$$P_p\left(R_n > \frac{(1+\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$

9. (continuation) Show that, for $\epsilon > 0$,

$$P_p\left(R_n < \frac{(1-\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$

   [This is the famous problem of the longest run of heads in n tosses of a coin.]

10. The following random 'dynamical system' has acquired a certain notoriety. At each vertex of $\mathbb{L}^2$ we place a mirror. These mirrors are plane and double-backed, and are placed such that light incident along an axis is reflected along a perpendicular axis. Each mirror has two possible configurations, labelled in the natural way NW and NE, and we stipulate that, for each vertex x, the mirror at x is equally likely to be NW as NE (and that mirrors at different vertices have independent orientations). We light a candle at the origin, and let C be the set of illuminated vertices. Show that $P(|C| < \infty) = 1$. [*Hint*. Use the fact that $\theta(\frac{1}{2}) = 0$ for bond percolation on the square lattice.]