Interacting particle systems III

1. Let ϕ be a random-cluster measure on a finite graph $G = (V, E)$ with parameters p and q. Prove that
$$\frac{d}{dp}\phi(A) = \frac{1}{p(1-p)}\Big\{\phi(N1_A) - \phi(N)\phi(A)\Big\}$$
for any event A, where N is the number of open edges of a configuration ω and 1_A is the indicator function of the event A. [The expression $\phi(Z)$ denotes the mean of Z under ϕ.]
2. (continuation) Show that ϕ satisfies the FKG inequality if $q \geq 1$, in that $\phi(A \cap B) \geq \phi(A)\phi(B)$ for increasing events a, b, but does not generally have this property when $q < 1$.
3. Show that random-cluster measures $\phi_{p,q}$ do not generally satisfy the BK inequality if $q > 1$. That is, find a finite graph G and increasing events A, B such that $\phi_{p,q}(A \circ B) > \phi_{p,q}(A)\phi_{p,q}(B)$.
4. (Important research problem, hard if true) Prove that random-cluster measures satisfy the BK inequality if $q < 1$.
5. Let $\phi_{p,q}$ be the random-cluster measure on a finite connected graph $G = (V, E)$. Show, in the limit as $p, q \to 0$ in such way that $q/p \to 0$, that $\phi_{p,q}$ converges weakly to the uniform spanning tree measure UST on G. Identify the corresponding limit as $p, q \to 0$ with $p = q$. Explain the relevance of these limits to the previous question.
6. **Comparison inequalities.** Use Holley's theorem to prove the following 'comparison inequalities' for a random-cluster measure $\phi_{p,q}$ on a finite graph.
$$\begin{aligned} \phi_{p',q'} &\leq_{\text{st}} \phi_{p,q} \qquad \text{if } q' \geq q,\ q' \geq 1,\ p' \leq p, \\ \phi_{p',q'} &\geq_{\text{st}} \phi_{p,q} \qquad \text{if } q' \geq q,\ q' \geq 1,\ \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}. \end{aligned}$$
7. Use the comparison inequalities of the previous question to prove that the critical point $p_c(q)$ of the random-cluster model on $\mathbb{Z}^d$ satisfies
$$p_c(1) \leq p_c(q) \leq \frac{qp_c(1)}{1+(q-1)p_c(1)}, \quad q \geq 1.$$
In particular, $0 < p_c(q) < 1$ if $q \geq 1$ and $d \geq 2$.
8. Each point of the square lattice is occupied, at each time t, by either a benign or a malignant cell. Benign cells invade their neighbours, each neighbour being invaded at rate β, and similarly malignant cells invade their neighbours at rate μ. Suppose there is exactly one malignant cell at time 0, and let $\kappa = \mu/\beta \geq 1$. Show that the malignant cells die out with probability κ^{-1}.
9. Let $\Omega = \{0, 1\}^E$ where E is finite, let P be a probability measure on Ω and $A \subseteq \Omega$. Show that $P(A)$ may be expressed as a linear combination of certain $P(A_i)$, $i \geq 1$, where the A_i are increasing events. [This fact is useful in proving the existence of infinite-volume limits in the presence of the FKG inequality.]

10. **Duality for voter model.** Let ν_t^A be the set of 1's in a voter model in which the initial set of 1's is A. Let R_t^B be the positions of a system of coalescing random walks which jump at rate 1 and which begin at points in B. Show that

$$P(\nu_t^A \cap B \neq \varnothing) = P(R_t^B \cap A \neq \varnothing).$$

11. **Duality for exclusion process.** Let $G = (V, E)$ be a finite or countably infinite *regular* graph. Particles move about the vertices of the graph according to the following rules. Each vertex may be occupied by no more than one particle at any given time. Each particle tries to jump at rate 1. When it jumps, it does so to a randomly chosen neighbour of its current position. If this target vertex is already occupied, then the jump does not take place. Let η_t be the configuration in the exclusion process at time t. Let A_t be the positions of the points in an exclusion process at time t, where $|A_0| < \infty$. [We distinguish between the general process η_t and a process A_t having only finitely many particles.] Show that

$$P^\eta(\eta_t \equiv 1 \text{ on } A) = P^A(\eta \equiv 1 \text{ on } A_t),$$

where P^κ denotes the probability measure governing the appropriate process conditional on starting in the state κ.
Deduce that, for all $0 \leq \rho \leq 1$, product measure with intensity ρ is invariant for the exclusion process.