Interacting particle systems I

1. Let $G = (V, E)$ be a finite connected graph with positive edge weights $(w_e : e \in E)$. Show, in the notation of the lectures that

$$i_{ab} = \frac{N^*(s, a, b, t) - N^*(s, b, a, t)}{N^*}$$

constitutes a unit flow through G from s to t satisfying Kirchhoff's laws.

2. (continuation) Let $G = (V, E)$ be finite and connected with given conductances $(w_e : e \in E)$, and let $(x_v : v \in V)$ be reals satisfying $\sum_v x_v = 0$. Show that there exists a solution i to Kirchhoff's laws, viewed as two laws concerning current flow, such that the current leaving the network at each v satisfies $i_{v\infty} = x_v$.

3. Prove the series and parallel laws for electrical resistances, and also the star/delta law.

4. Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave in that

$$\tfrac{1}{2}\big(R(r_1) + R(r_2)\big) \le R\big(\tfrac{1}{2}(r_1 + r_2)\big).$$

5. Let G be an infinite connected graph, and let $\partial\Lambda_n$ be the set of vertices distance n from the vertex labelled O. With E_n the number of edges joining $\partial\Lambda_n$ to $\partial\Lambda_{n+1}$, show that random walk on G is recurrent if $\sum_n E_n^{-1} = \infty$.

6. (continuation) Assume that G is 'spherically symmetric' in that: for all n, for all $x, y \in \partial\Lambda_n$, there exists a graph automorphism which fixes O and maps x to y. Show that random walk on G is transient if $\sum_n E_n^{-1} < \infty$.

7. Let G be a finite network with positive conductances $(w_e : e \in E)$, and let a, b be distinct vertices. Let i_{xy} denote the current along an edge from x to y when a unit current flows from the source vertex a to the sink vertex b. Run the associated Markov chain, starting at a, until it reaches b for the first time, and let u_{xy} be the mean of the net number of transitions of the chain between x and y. Transitions from x to y count positive, and from y to x negative, so that u_{xy} is the mean number of transitions from x to y, minus the mean number from y to x. Show that $i_{xy} = u_{xy}$.

8. Let G be a finite connected graph, and let T be a randomly chosen spanning tree of G, each such tree being equally likely. Let $e = \langle x, y\rangle$ be an edge of G. Show that $P(e \in T) = R(e)$, where $R(e)$ is the effective resistance of G between the terminals x and y, with each edge of G representing a unit resistor. Deduce that

$$P(e, f \in T) \le P(e \in T)P(f \in T)$$

for distinct edges e, f.

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