Interacting particle systems, Handout I

Consider the d-dimensional lattice $\mathbb{Z}^d$, with $(\Omega, \mathcal{F})$ given as usual, and let P be a product measure on this space with some constant density p. Let τ be a translation of $\mathbb{Z}^d$, so that τ acts on Ω in the usual way by $\tau\omega(e) = \omega(\tau^{-1}e)$.

Let N be a random variable on $(\Omega, \mathcal{F})$ which is τ-invariant in that $N(\omega) = N(\tau\omega)$ for $\omega \in \Omega$. We show below that N is necessarily P-almost-surely constant, in that there exists $n \in \mathbb{R}$ such that $P(N = n) = 1$.

It suffices to show that, for any τ-invariant event A (in that $A = \tau A$), either $P(A) = 1$ or $P(A) = 1$. The claim above will follow by applying this to the events $\{N \le x\}$ for $x \in \mathbb{R}$.

Let A be a τ-invariant event, and let $\epsilon > 0$. Remembering that $\mathcal{F}$ is the σ-field generated by the finite-dimensional cylinders, the following is a basic fact of measure theory. There exists a finite set F of edges, and an event A_F defined in terms of the states of edges in F only, such that $P(A \bigtriangleup A_F) \le \epsilon$, where $\bigtriangleup$ denotes symmetric difference.

By translation by τ^k, $P(\tau^k A \bigtriangleup \tau^k A_F) \le \epsilon$. Now $\tau^k A = A$, so that $P(A \bigtriangleup \tau^k A) \le \epsilon$, and therefore

$$P(A_F \bigtriangleup \tau^k A_F) \le 2\epsilon.$$

We choose k such that $F \cap \tau^k F = \varnothing$. Since P is a product measure,

$$P(A_F \cap \tau^k A_F) = P(A_F)P(\tau^k A_F) = P(A_F)^2.$$

Finally,

$$\begin{aligned}
2\epsilon &\ge P(A_F \bigtriangleup \tau^k A_F) \\
&= P(A_F) + P(\tau^k A_F) - 2P(A_F \cap \tau^k A_F) \\
&= 2\big[P(A_F) - P(A_F)^2\big] \\
&\ge 2\big[P(A) - \epsilon - (P(A) + \epsilon)^2\big].
\end{aligned}$$

Let $\epsilon \downarrow 0$ to find that $P(A) \le P(A)^2$, so that $P(A)$ equals either 0 or 1.

Note: This is a standard argument, and is stated often as 'the product measure P is ergodic'. 'Recall' that a measure is called ergodic (or in this case, ergodic with respect to τ) if every τ-invariant random variable is μ-almost-surely constant. The conclusion is of course related to the Kolmogorov zero–one law, which states that every tail event is trivial with respect to the product measure P.

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