ADVANCED PROBABILITY

Comments on Example Sheet 3

1. Let $\epsilon > 0$. The event $I = \{Y - Y' > \epsilon\}$ lies in $\mathcal{G}$, whence $E(Y1_I) = E(X1_I) = E(Y'1_I)$. Therefore $0 = E((Y - Y')1_I) \geq \epsilon P(I)$ whence $P(I) = 0$. Hence

$$P(Y > Y') = P\Big(\bigcup_{k\geq 1}\{Y - Y' > k^{-1}\}\Big) = 0,$$

and similarly $P(Y < Y') = 0$.

2. (a) Suppose that $\{X_n\}$ is Cauchy convergent in probability. For each positive integer k, there exists n_k such that

$$P\big(|X_n - X_m| \geq 2^{-k}\big) < 2^{-k} \qquad \text{for } n, m \geq n_k.$$

The sequence (n_k) may not be increasing, and we work instead with the sequence defined by $N_1 = n_1$, $N_{k+1} = \max\{N_k + 1, n_{k+1}\}$. We have that

$$\sum_k P\big(|X_{N_{k+1}} - X_{N_k}| \geq 2^{-k}\big) < \infty,$$

whence, by the first Borel–Cantelli lemma, a.s. only finitely many of the events $\{|X_{N_{k+1}} - X_{N_k}| \geq 2^{-k}\}$ occur. Therefore, the expression

$$X = X_{N_1} + \sum_{k=1}^{\infty}(X_{N_{k+1}} - X_{N_k})$$

converges absolutely on an event C having probability one. Define $X(\omega)$ accordingly for $\omega \in C$, and $X(\omega) = 0$ for $\omega \notin C$. We have, by the definition of X, that $X_{N_k} \to X$ a.s. as $k \to \infty$. Finally, we 'fill in the gaps'. For $\epsilon > 0$,

$$P\big(|X_n - X| > \epsilon\big) \leq P\big(|X_n - X_{N_k}| > \tfrac{1}{2}\epsilon\big) + P\big(|X_{N_k} - X| > \tfrac{1}{2}\epsilon\big) \to 0$$

as $n, k \to \infty$, where we are using the assumption that $\{X_n\}$ is Cauchy convergent in probability.

(b) Let $\{X_n : n \geq 1\}$ be Cauchy convergent in mean square, that is, $E\{|X_n - X_m|^2\} \to 0$ as $m, n \to \infty$. By Chebyshev's inequality, $\{X_n : n \geq 1\}$ is Cauchy convergent in probability, and therefore converges in probability to some limit X (proof as above). As above, there exists a subsequence $\{X_{n_k} : k \geq 1\}$ which converges to X almost surely. Now, as $n \to \infty$,

$$E\{|X_n - X|^2\} = E\left\{\liminf_{k\to\infty} |X_n - X_{n_k}|^2\right\} \leq \liminf_{k\to\infty} E\{|X_n - X_{n_k}|^2\} \to 0,$$

where we have used Fatou's lemma and Cauchy convergence in mean square. Therefore $X_n \to X$ in mean square. That $E(X^2) < \infty$ follows by Minkowski's inequality thus:

$$E(X^2)^{1/2} \leq E\{|X_n - X|^2\}^{1/2} + E(X_n^2)^{1/2} \qquad \text{for any fixed } n.$$

3. See Grimmett and Stirzaker (1992), Section 7.9.

4. The important step lies in the proof of Kolmogorov's Inequality. At one point one needs the martingale property in place of independence, namely:

$$E\big(S_k 1_k (S_n - S_k)\big) = E\big(S_k 1_k E(S_n - S_k \mid \mathcal{F}_k)\big)$$

where $\mathcal{F}_k = \sigma(X_1, X_2, \ldots, X_k)$, and I_k is the indicator function of a certain event in $\mathcal{F}_k$. Another proof of the (Doob–)Kolmogorov inequality uses the so called 'maximal inequality' for martingales: see GS, p. 470.

5. We have that

$$(*)\qquad Z_{n+1} = F_{1,n} + F_{2,n} + \cdots + F_{Z_n,n}$$

where $F_{i,j}$ is the family-size of the ith member of the jth generation. Therefore $E(Z_{n+1}) = \mu E(Z_n)$, and so $E(A_{n+1} \mid A_n) = A_n$ as required.

Secondly, $\eta_n = P(Z_n = 0)$ satisfies

$$\eta_n = E\big(P(Z_n = 0 \mid Z_1)\big) = E(\eta_{n-1}^{Z_1}) = G(\eta_{n-1}).$$

Letting $n \to \infty$ and using the continuity of G on $[0,1]$, we deduce that $\eta = \lim_{n\to\infty} \eta_n$ satisfies $\eta = G(\eta)$. Now

$$E(B_{n+1} \mid Z_n) = E(\eta^{F_{1,n}+F_{2,n}+\cdots} \mid Z_n) = G(\eta)^{Z_n} = \eta^{Z_n}.$$

For the second part, A_n is a non-negative martingale, whence there exists A such that $A_n \to A$ a.s. By conditioning on Z_1, we find that the characteristic function ϕ_n of A_n satisfies

$$\phi_n(t) = G\big(\phi_{n-1}(tEZ_{n-1}/EZ_n)\big) = G\big(\phi_{n-1}(t/\mu)\big).$$

Since $A_n \to A$ a.s., we have that $\phi_n(t) \to \phi(t)$ where ϕ is the c.f. of A. The claim follows by the continuity of G. It is quite an art to extract information about ϕ from this functional equation.

6. Y is a martingale since Y_n is the sum of independent variables with zero means. Also $\sum_1^\infty P(Z_n \neq 0) = \sum_1^\infty n^{-2} < \infty$, implying by the Borel–Cantelli lemma that $Z_n = 0$ except for finitely many values of n (a.s.); therefore the partial sum Y_n converges a.s. as $n \to \infty$ to some finite limit.

It is easily seen that $a_n = 5a_{n-1}$ and therefore $a_n = 8 \cdot 5^{n-2}$, if $n \geq 3$. It follows that $|Y_n| \geq \frac{1}{2}a_n$ if and only if $|Z_n| = a_n$. Therefore

$$E|Y_n| \geq \tfrac{1}{2}a_n P(|Y_n| \geq \tfrac{1}{2}a_n) = \tfrac{1}{2}a_n P(|Z_n| = a_n) = \frac{a_n}{2n^2}$$

which tends to infinity as $n \to \infty$.

7. With $I_k = I_{\{Y_k=0\}}$, we have that

$$\begin{aligned} E(Y_n \mid \mathcal{F}_{n-1}) &= E\Big(X_n I_{n-1} + nY_{n-1}|X_n|(1 - I_{n-1}) \,\Big|\, \mathcal{F}_{n-1}\Big) \\ &= I_{n-1}E(X_n) + nY_{n-1}(1 - I_{n-1})E|X_n| = Y_{n-1} \end{aligned}$$

since $E(X_n) = 0$, $E|X_n| = n^{-1}$. Also $E|Y_n| \leq E\big\{|X_n|(1+n|Y_{n-1}|)\big\}$ and $E|Y_1| < \infty$, whence $E|Y_n| < \infty$. Therefore $(Y, \mathcal{F})$ is a martingale.

Now $Y_n = 0$ if and only if $X_n = 0$. Therefore $P(Y_n = 0) = P(X_n = 0) = 1 - n^{-1} \to 1$ as $n \to \infty$, implying that $Y_n \to 0$ in probability. On the other hand, $\sum_n P(X_n \neq 0) = \infty$, and therefore $P(Y_n \neq 0 \text{ i.o.}) = 1$ by the second Borel–Cantelli lemma. However, Y_n takes only integer values, and therefore Y_n does not converge to 0 a.s. The martingale convergence theorem is inapplicable since $\sup_n E|Y_n| = \infty$.

8. Let $\mathcal{F}_n = \sigma(R_1, R_2, \ldots, R_n)$.

(a) $0 \leq Y_n \leq 1$, and Y_n is $\mathcal{F}_n$-measurable. Also

$$E(R_{n+1} \mid R_n) = R_n + \frac{R_n}{n+r+b},$$

whence Y_n satisfies $E(Y_{n+1} \mid \mathcal{F}_n) = Y_n$. Therefore $\{Y_n : n \geq 0\}$ is a uniformly integrable martingale, and therefore converges a.s. and in mean.

(b) In order to apply the optional stopping theorem, it suffices that $P(T < \infty) = 1$ (since Y is uniformly integrable). However $P(T > n) = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = (n+1)^{-1} \to 0$. Using that theorem, $E(Y_T) = E(Y_0)$, which is to say that $E\{T/(T+2)\} = \frac{1}{2}$, and the result follows.

(c) Apply the maximal inequality for martingales.

9. Since the game is fair, $E(S_{n+1} \mid S_n) = S_n$. Also $|S_n| \le 1 + 2 + \cdots + n < \infty$. Therefore S_n is a martingale. The occurrence of the event $\{N = n\}$ depends only on the outcomes of the coin-tosses up to and including the nth; therefore N is a stopping time.

A tail appeared at time $N-3$, followed by three heads. Therefore the gamblers $G_1, G_2, \dots, G_{N-3}$ have forfaited their initial capital by time N, while G_{N-i} has had $i+1$ successful rounds for $0 \le i \le 2$. Therefore $S_N = N - (p^{-1} + p^{-2} + p^{-3})$, after a little calculation. It is easy to check that N satisfies the conditions of the optional stopping theorem, and it follows that $E(S_N) = E(S_0) = 0$, which is to say that $E(N) = p^{-1} + p^{-2} + p^{-3}$.

In order to deal with HTH, the gamblers are re-programmed to act as follows. If they win on their first bet, they bet their current fortune on *tails*, returning to heads thereafter. In this case, $S_N = N - (p^{-1} + p^{-2}q^{-1})$ where $q = 1-p$ (remember that the game is fair), and therefore $E(N) = p^{-1} + p^{-2}q^{-1}$.

10. Certainly $0 \le X_n \le 1$, and in addition X_n is measurable with respect to the σ-field $\mathcal{F}_n = \sigma(R_1, R_2, \dots, R_n)$. Also $E(R_{n+1} \mid R_n) = R_n - R_n/(52-n)$, whence $E(X_{n+1} \mid \mathcal{F}_n) = X_n$. Therefore X_n is a martingale.

A strategy corresponds to a stopping time. If the player decides to call at the stopping time T, he wins with (conditional) probability X_T, and therefore $P(\text{wins}) = E(X_T)$, which equals $E(X_0)$ $(= \frac{1}{2})$ by the optional stopping theorem.

[There is also a 'trivial' solution which uses no technology.]

11. Assume that $t > 0$ and $M(t) = 1$. Then $Y_n = e^{tS_n}$ defines a positive martingale (with mean 1) with respect to $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$. By the maximal inequality,

$$P\left(\max_{1\le k\le n} S_k \ge x\right) = P\left(\max_{1\le k\le n} Y_k \ge e^{tx}\right) \le e^{-tx}E(Y_n),$$

and the result follows by taking the limit as $n \to \infty$.

12. We have that

$$\begin{aligned} \{T_1 + T_2 = n\} &= \bigcup_{k=0}^{n} \big(\{T_1 = k\} \cap \{T_2 = n-k\}\big), \\ \big\{\max\{T_1, T_2\} \le n\big\} &= \{T_1 \le n\} \cap \{T_2 \le n\}, \\ \big\{\min\{T_1, T_2\} \le n\big\} &= \{T_1 \le n\} \cup \{T_2 \le n\}. \end{aligned}$$

Each event on the right-hand side lies in $\mathcal{F}_n$.

13. We have that $S_n = X_1 + X_2 + \cdots + X_n$ satisfies

$$S_{N(t)} < t \le S_{N(t)+1}.$$

By Wald's equation, $t \le E(S_{N(t)+1}) = \mu E(N(t) + 1)$, whence the claim follows. For the second part, a little truncation of the X_j may be useful; see Grimmett and Stirzaker, p. 394.