ADVANCED PROBABILITY

Comments on Example Sheet 2

1. This is a 'simple' computation.

2. If X_n has distribution function F_n, then

$$P(|X_n| \le a) \le \frac{1}{h(a)} \int \phi(x) dF_n(x)$$

where $h(a) = \inf\{\phi(x) : |x| \ge a\}$. Now $h(a) \to \infty$ as $a \to \infty$, and the integral is bounded uniformly in n.

3. Use the Skorohod representation, as in Question 4.

4. Use the Skorohod representation. Find random variables $Y, Y_1, Y_2, \dots$ having the same marginal distributions as $X, X_1, X_2, \dots$, and such that $Y_n \to Y$ a.s. Since g is continuous, we have that $g(Y_n) \to g(Y)$ a.s., whence $g(Y_n) \Rightarrow g(Y)$. Therefore $g(X_n) \Rightarrow g(X)$.

5. See Grimmett and Stirzaker (1992), p. 283.

6. The characteristic function of X_n is

$$\begin{aligned}\prod_{j=1}^{n} E(e^{itY_j 4^{-j}}) &= \prod_{j=1}^{n} \left\{ \frac{1}{4} \cdot \frac{1 - e^{it4^{-j+1}}}{1 - e^{it4^{-j}}} \right\} \\ &= \frac{1 - e^{it}}{4^n (1 - e^{it4^{-n}})} \to \frac{1 - e^{it}}{it},\end{aligned}$$

the uniform characteristic function. Note that X_n is monotone, and therefore converges a.s.

7. With $g(u) = u/(1+u)$,

$$P(|X_n| > \epsilon) = P\big(g(|X_n|) > g(\epsilon)\big) \le \frac{1}{g(\epsilon)} E\big(g(|X_n|)\big).$$

Also

$$\begin{aligned}E\big(g(|X_n|)\big) &\le g(\epsilon) + 1 \cdot P(|X_n| > \epsilon) \\ &\to g(\epsilon) \quad \text{if} \quad X_n| \xrightarrow{P} 0.\end{aligned}$$

8. Let $X_1, X_2, \dots$ be independent with the Poisson distribution, mean λ. Then $S_n = X_1 + \cdots + X_n$ has the Poisson distribution, mean $n\lambda$. Using the Law of Large Numbers,

$$\sum_{k=0}^{\infty} g(k/n) \frac{(n\lambda)^k}{k!} e^{-n\lambda} = E\big(g(S_n/n)\big) \to g(\lambda)$$

as $n \to \infty$, since $S_n/n \Rightarrow \lambda$.

9. (a) Density function $f(x) = (1 - \cos x)/(\pi x^2)$.

(b) $\cos t = \frac{1}{2}(e^{it} + e^{-it})$. Let X satisfy $P(X = 1) = P(X = -1) = \frac{1}{2}$.

(c) $\phi(t) = (1+t)^{-4}$ satisfies $\phi'(0) = \phi''(0) = 0$. If ϕ were the characteristic function of X, then $\mathrm{var}\,(X) = 0$; but then X has characteristic function 1, a contradiction.

(d) Take another look at part (a).

10. Related to Question 8. This time use the CLT.

11. Let ϕ_n be characteristic function of the $N(0,n)$ distribution, i.e., $\phi_n(t) = e^{-t^2 n/2}$.

12. Since f is continuous on a closed interval, it is bounded: $|f(y)| \le c$ for all $y \in [0,1]$ for some c. Furthermore f is uniformly continuous on $[0,1]$, which is to say that, if $\epsilon > 0$, there exists δ (> 0), such that $|f(y) - f(z)| < \epsilon$ if $|y - z| \le \delta$. Let $A = \{|n^{-1}S_n - x| > \delta\}$, and $Z = f(x) - f(n^{-1}S_n)$. With the above choice of ϵ, δ, we have that $|E(ZI_{A^c})| < \epsilon$, and

$$|E(ZI_A)| \le 2c\mathbb{P}(A) \le 2c \cdot \frac{x(1-x)}{n\delta^2}$$

by Chebyshev's inequality. Therefore

$$|E(Z)| \le |E(ZI_A)| + |E(ZI_{A^c})| \le 2c \cdot \frac{x(1-x)}{n\delta^2} < \epsilon + \frac{2c}{n\delta^2},$$

which is less than 2ϵ for values of n exceeding $2c/(\epsilon\delta^2)$.

13. Either do it long-hand (not very long!), or use the Skorohod representation.