ADVANCED PROBABILITY

Comments on Example Sheet 1

1. Use the fact that

$$\bigcap_i A_i = \overline{\bigcup_i \overline{A_i}}$$

where $\bar{A}$ denotes the complement of A.

2. To check that $\mathcal{G}$ is a field is easy. Be careful about what you are being asked to prove next, namely that, if $A_1, A_2, \dots$ are disjoint members of $\mathcal{G}$ such that $A = \bigcup_i A_i \in \mathcal{G}$, then $\mu(A) = \sum_1^\infty \mu(A_i)$. A proof may be found in the listed book by Billingsley. This result enables a *unique* extension of μ to the Borel field of S, called *Lebesgue measure*.

3. Either use induction, or use the facts that

$$1_{\bigcup_i A_i} = 1 - \prod_i (1 - 1_{A_i}),$$

$$1_{\bigcap_i B_i} = \prod_i 1_{B_i}.$$

Expand the first relation, re-express the terms using the second, and apply μ. We need the condition $\mu(\bigcup A_i) < \infty$.

4. Z is a random variable iff $Z^{-1}(B) \in \mathcal{F}$ for all Borel B, and it suffices that either (i) or (ii) hold:

(i) $\{Z \le x\} \in \mathcal{F}$ for all x,
(ii) $\{Z < x\} \in \mathcal{F}$ for all x.

Now, with $\mathbb{Q}$ the rationals,

(a) $\{-X < x\} = \{X \ge -x\} = \overline{\{X \le -x\}} \in \mathcal{F}$,
(b) $\{X + Y < x\} = \bigcup_{q \in \mathbb{Q}} (\{X < q\} \cap \{Y < x - q\})$, a countable union,
(c) $\{\max\{X, Y\} \le x\} = \{X \le x\} \cap \{Y \le y\}$,
(d) $\{\min\{X, Y\} \le x\} = \{X \le x\} \cup \{Y \le y\}$.

5. $\{\limsup X_n < x\} = \bigcup_n \bigcap_{m \ge n} \{X_m < x\}$, and $\liminf X_n = -\limsup\{-X_n\}$.

6. Let $J \subseteq \{1, 2, \dots, n\}$. If $1 \notin J$, then nothing new to prove. Suppose $1 \in J$. Then, with $B = \bigcup_{i \in J \setminus \{1\}} A_i$,

$$\begin{aligned} P(\overline{A_1} \cap B) &= P(B) - P(A_1 \cap B) \\ &= P(B) - P(A_1)P(B) = P(\overline{A_1})P(B). \end{aligned}$$

Also $P(B) = \prod_{i \in J \setminus \{1\}} P(A_i)$.

7. (a) Check that

$$\sum_{i=1}^\infty 1_{\{X \ge i\}} \le X \le \sum_{i=0}^\infty 1_{\{X \ge i\}}$$

by considering the case when $n \le X < n + 1$. Now

$$E\Big(\sum_1^\infty 1_{\{X \ge i\}}\Big) = \sum_1^\infty P(X \ge i)$$

by monotone convergence, and the result follows.

(b) $P(X_n \ne Y_n) = P(X_n > n) = P(X_1 > n)$, whence, by the Borel–Cantelli lemmas, the given dichotomy depends on the divergence or convergence of $\sum_n P(X_1 > n)$. By part (a), this is equivalent to the divergence or convergence of $E(X_1)$.

8. $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0$ if X, Y are independent. For $t \geq 0$,

$$\phi(t) = E\big((X+tY)^2\big) \geq 0,$$

whence the quadratic

$$E(X^2) + 2tE(XY) + t^2E(Y^2) = 0$$

has at most one real zero. Therefore the discriminant is non-positive, implying that

$$\big(E(XY)\big)^2 \leq E(X^2)E(Y^2).$$

This implies the required inequality by substituting $X - EX$ and $Y - EY$. Take care with parentheses.

9. Let $A_n(\epsilon) = \{|Z_n - Z| > \epsilon\}$. By BC1, $P(A_n(\epsilon) \text{ i.o.}) = 0$ for all $\epsilon > 0$. Now

$$\{Z_n \to Z\} = \bigcup_{\epsilon>0} \{A_n(\epsilon) \text{ i.o.}\}.$$

Now $A_n(\epsilon) \subseteq A_n(\epsilon')$ if $\epsilon \leq \epsilon'$, whence

$$P\left(\bigcup_{\epsilon>0}\{A_n(\epsilon) \text{ i.o.}\}\right) = P\left(\bigcup_{k\geq 1}\{A_n(k^{-1}) \text{ i.o.}\}\right) = \lim_{k\to\infty} P\Big(A_n(k^{-1}) \text{ i.o.}\Big) = 0.$$

10. By the Cauchy–Schwarz inequality

$$E(S_n^4) = \sum_1^n E(X_1^4) + 12\sum_{i<j} E(X_i^2X_j^2) \leq nM + 6n(n-1)M \leq 6n^2M,$$

whence, by Markov's inequality

$$\sum_n P\left(\frac{|S_n|}{n} > \epsilon\right) \leq \sum_n \frac{E(S_n^4)}{n^4\epsilon^4} < \infty$$

if $\epsilon > 0$. Now apply previous problem.

11. The easiest way is to use Jensen's inequality, which states that $E(f(X)) \geq f(E(X))$ for any convex function f and random variable X such that the expectations exist. Apply this with $f(u) = u^{r/s}$ for $u \geq 0$, $X = |Z|^s$.

12. With $F(x) = P(X \leq x)$, we have that

$$F(y) = F(x) + P(x < X \leq y) \geq F(x) \quad \text{if } x \leq y.$$

Also, using the monotonicity,

$$\begin{aligned}\lim_{h\downarrow 0} F(x+h) &= \lim_{k\to\infty} F\left(x + \frac{1}{k}\right) = \lim_k P\left(X \leq x + \frac{1}{k}\right)\\ &= P\left(\lim_k\left\{X \leq x + \frac{1}{k}\right\}\right) = P(X \leq x)\end{aligned}$$

by the continuity of P. Similarly, as $x \downarrow -\infty$,

$$F(x) \geq F(\lfloor x\rfloor) \to P\Big(\lim_{k\to\infty}\{X \leq -k\}\Big) = P(\varnothing) = 0,$$

and similarly $F(x) \to 1$ as $x \to \infty$.

Conversely, take $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, and define P by $P((a,b]) = F(b) - F(a)$. This definition of P extends to a unique probability measure on $(\Omega, \mathcal{B})$, as in Problem 2. Now define $X : \mathbb{R} \to \mathbb{R}$ by $X(x) = x$.