MATHEMATICAL TRIPOS: PART IA (Lent 2002)

Probability — Example Sheet 4 (out of 4) **GRG**

1. Alice and Bob agree to meet in the Copper Kettle after their Saturday lectures. They arrive at times that are independent and uniformly distributed between 12.00 and 1.00 pm. Each is prepared to wait 10 minutes before leaving. Find the probability they meet.

2. A stick is broken in two places, independently uniformly distributed along its length. What is the probability that the three pieces will make a triangle?

3. The radius of a circle is exponentially distributed with parameter λ. Determine the probability density function of the area of the circle.

4. The random variables X and Y are independent and exponentially distributed with parameters λ and μ respectively. Find the distribution of $\min\{X, Y\}$, and the probability that X exceeds Y.

5. How large a random sample should be taken from a normal distribution in order for the probability to be at least 0.99 that the sample mean will be within one standard deviation of the mean of the distribution?

[*Hint:* $\Phi(2.58) = 0.995$.]

6. The random variable X has a *log-normal distribution* if $Y = \log X$ is normally distributed. If $Y \sim N(\mu, \sigma^2)$, calculate the mean and variance of X. (The log-normal distribution is sometimes used to represent the size of small particles after a crushing process, or as a model for future commodity prices. Why?)

7. X and Y are independent random variables, each distributed normally, as $N(0, 1)$. Show that, for any fixed θ, the random variables

$$X \cos\theta + Y \sin\theta, \qquad -X \sin\theta + Y \cos\theta$$

are independent and find their distributions.

8. The random variables X and Y are independent and exponentially distributed, each with parameter λ. Show that the random variables $X + Y$ and $X/(X + Y)$ are independent and find their distributions.

9. A shot is fired at a circular target. The vertical and horizontal coordinates of the point of impact (taking the centre of the target as origin) are independent random variables, each distributed normally $N(0, 1)$.

(i) Show that the distance of the point of impact from the centre has p.d.f. $re^{-r^2/2}$ for $r \geq 0$.
(ii) Show that the mean of this distance is $\sqrt{\pi/2}$, that the median is $\sqrt{\log 4}$, and that the mode is 1.

10. A radioactive source emits particles in a random direction (with all directions being equally likely). It is held at a distance d from a vertical infinite plane photographic plate.

(i) Show that, given the particle hits the plate, the horizontal coordinate of its point of impact (with the point nearest the source as origin) has p.d.f. $d/\pi(d^2 + x^2)$. (This distribution is known as the Cauchy distribution).
(ii) Can you compute the mean of this distribution?

11. A random sample is taken in order to find the proportion of Labour voters in a population. Find a sample size such that the probability of a sampling error less than 0.04 will be 0.99 or greater.

12. The random variables $Y_1, Y_2, \ldots, Y_n$ are independent, with $\mathbb{E}Y_i = \mu_i$, $\operatorname{Var} Y_i = \sigma^2$, $1 \leq i \leq n$. For constants $a_i, b_i, 1 \leq i \leq n$, show that

$$\operatorname{cov}\left(\sum_i a_i Y_i, \sum_i b_i Y_i\right) = \sigma^2 \sum_i a_i b_i.$$

Prove that if $Y_1, Y_2, \ldots, Y_n$ are independent normal random variables, then $\sum_i a_i Y_i$ and $\sum_i b_i Y_i$ are independent if and only if $\sum_i a_i b_i = 0$.

13. Buffon's needle. You wish to determine π by repeatedly dropping a straight pin of length l ($< L$) onto a floor marked with parallel lines spaced L apart. Estimate how closely you could determine the value of π by devoting 50 years to full-time pin dropping. What pin length, l, would you prefer?

14. (i) Simulate 100 random samples of size $n = 2$ from a uniform distribution on $[0, 1]$, record the sample mean for each such sample, and plot a histogram of the resulting sample means. What does it look like? What happens if you repeat the exercise with $n = 10$?
(ii) Repeat part (i), but replace the uniform distribution on $[0, 1]$ with the Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty\,.$$

Comment on any qualitative differences between these results and those obtained in part (i). [*Hint:* If $U \sim U[0, 1]$, then $X = \tan(\pi(U - \frac{1}{2}))$ has the density f above.]

Problems

15. A random sample of size $2n + 1$ is taken from the uniform distribution on $[0, 1]$. Find the distribution of the sample median.

16. Suppose that n items are being tested simultaneously and that the items have independent lifetimes, each exponentially distributed with parameter λ. Determine the mean and variance of the length of time until r items have failed.

17. (i) X and Y are independent random variables, with continuous symmetric distributions, with p.d.f.s f and g respectively. Show that the p.d.f. of $Z = X/Y$ is

$$h(a) = 2\int_0^\infty x f(ax) g(x) dx\,.$$

(ii) X and Y are independent random variables distributed $N(0, \sigma^2)$ and $N(0, \tau^2)$. Show that X/Y has p.d.f. $f(x) = d/\pi(d^2 + x^2)$, where $d = \sigma/\tau$.

18. $X_1, X_2, \ldots$ are independent Cauchy random variables, each with p.d.f.

$$f(x) = \frac{d}{\pi(d^2 + x^2)}.$$

Show that $n^{-1}(X_1 + X_2 + \cdots + X_n)$ has the same distribution as X_1.
[*Hint:* Try first the case $d = 1, n = 2$, and use the identity

$$m\int_{-\infty}^{+\infty} \{(1+y^2)[m^2 + (x-y)^2]\}^{-1} dy = \pi(m+1)\{(m+1)^2 + x^2\}^{-1}. \qquad]$$

Does this contradict the weak law of large numbers or the central limit theorem?

19. Derive the distribution of the sum of n independent random variables each having the Poisson distribution with parameter 1. Use the central limit theorem to prove that

$$e^{-n}\left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2} \quad \text{as } n \to \infty\,.$$

20. X_1,$X_2, \ldots$ form a sequence of independent random variables, each uniformly distributed on $(0, 1)$. Let

$$N = \min\{n : X_1 + X_2 + \ldots + X_n \geq 1\}$$

Show that $EN = e$. [*Hint:* calculate $P\{N \geq k\}$.]

21. If X, Y and Z are independent random variables each uniformly distributed on $(0, 1)$, show that $(XY)^Z$ is also uniformly distributed on $(0, 1)$.