MATHEMATICAL TRIPOS: PART IA (Lent 2002)

Probability — Example Sheet 3 (out of 4) **GRG**

Exercises

1. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. The geometric mean lies between the harmonic mean and the arithmetic mean:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} \le \left(\prod_{i=1}^{n} x_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} x_i\,.$$

The second inequality is the AM–GM inequality. Establish the first inequality.

2. Let X be a positive random variable taking only finitely many values. Show that

$$E\left(\frac{1}{X}\right) \ge \frac{1}{EX}\,,$$

and that the inequality is strict unless $P(X = EX) = 1$.

3. Let X be a random variable for which $EX = \mu$ and $E[(X-\mu)^4] = \beta_4$. Prove that

$$P(|X-\mu| \ge t) \le \frac{\beta_4}{t^4}.$$

4. How large a random sample should be taken from a distribution in order for the probability to be at least 0.99 that the sample mean will be within two standard deviations of the mean of the distribution? Use Chebyshev's inequality to determine a sample size that will be sufficient, whatever the distribution.

5. In a sequence of Bernoulli trials, X is the number of trials up to and including the ath success. Show that

$$P(X = r) = \binom{r-1}{a-1} p^a q^{r-a}, \qquad r = a, a+1, \ldots\,.$$

Verify that the probability generating function for this distribution is $G(t) = p^a t^a (1-qt)^{-a}$. Show that $EX = a/p$ and $\mathrm{var}X = aq/p^2$. Show how X can be represented as the sum of a independent random variables, all with the same distribution. Use this representation to derive again the mean and variance of X.

6. For a random variable X with mean μ and variance σ^2 define the function $V(x) = E[(X-x)^2]$. Express the random variable $V(X)$ in terms of μ, σ^2 and X, and hence show that $\sigma^2 = \frac{1}{2}E(V(X))$.

7. Let N be a non-negative integer-valued random variable with mean μ_1 and variance σ_1^2, and let $X_1, X_2, \ldots$ be identically distributed random variables, each with mean μ_2 and variance σ_2^2; furthermore, assume that $N, X_1, X_2, \ldots$ are independent. Calculate the mean and variance of the random variable $S_N = X_1 + \cdots + X_N$.

8. At time 0, a blood culture starts with one red cell. At the end of one minute, the red cell dies and is replaced by one of the following combinations with probabilities as indicated:

$$\text{2 red cells } \tfrac{1}{4}; \qquad \text{1 red, 1 white } \tfrac{2}{3}; \qquad \text{2 white } \tfrac{1}{12}.$$

Each red cell lives for one minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for one minute and dies without reproducing. Assume the individual cells behave independently.

(a) At time $n + \frac{1}{2}$ minutes after the culture began, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture dies out eventually?

9. (a) A mature individual produces offspring according to the probability generating function $G(s)$. Suppose we start with a population of k immature individuals, each of which grows to maturity with probability p and then reproduces, independently of the other individuals. Find the probability generating function of the number of (immature) individuals at the next generation.

(b) Find the probability generating function of the number of mature individuals at the next generation, given that there are k mature individuals in the parent generation.

10. Show that the distributions in (a) and (b) of (question **9**) have the same mean, but not necessarily the same variance.

11. A slot machine operates so that at the first turn the probability for the player to win is $\frac{1}{2}$. Thereafter the probability for the player to win is $\frac{1}{2}$ if he lost at the last turn, but is p $(< \frac{1}{2})$ if he won at the last turn. If u_n is the probability that the player wins at the nth turn, show that, provided $n > 1$,

$$u_n + (\tfrac{1}{2} - p)u_{n-1} = \tfrac{1}{2}.$$

Observe that this equation also holds for $n = 1$, if u_0 is suitably defined. Solve the equation, showing that

$$u_n = \frac{1 + (-1)^{n-1}(\frac{1}{2} - p)^n}{3 - 2p}.$$

12. A fair coin is tossed n times. Let u_n be the probability that the sequence of tosses never has 'head' followed by 'head'. Show that $u_n = \frac{1}{2}u_{n-1} + \frac{1}{4}u_{n-2}$. Find u_n, using the condition $u_0 = u_1 = 1$. Check that the value for u_2 is correct.

Problems

13. Let $b_1, b_2, \ldots, b_n$ be a rearrangement of the positive real numbers $a_1, a_2, \ldots, a_n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n\,.$$

14. Let $G(s) = 1 - \sqrt{1-s}$. Prove that $G(s)$ is a probability generating function and that its iterates are

$$G_n(s) = 1 - (1-s)^{2^{-n}} \quad \text{for } n = 1, 2, \ldots.$$

Find the mean m of the associated distribution and the extinction probability, $\eta = \lim_{n\to\infty} G_n(0)$, for a branching process with offspring distribution determined by G.

15. Let $(X_n)_{n\geq 0}$ be a branching process such that $X_0 = 1$, $EX_1 \equiv \mu$. If $Y_n = X_0 + X_1 + \cdots + X_n$, and $\Psi_n(s) = E(s^{Y_n})$ for $0 \leq s \leq 1$, prove that

$$\Psi_{n+1}(s) = s\phi(\Psi_n(s))\,,$$

where $\phi(s) \equiv Es^{X_1}$. Deduce that, if $Y = \sum_{n\geq 0} X_n$, then $\Psi(s) \equiv Es^Y$ satisfies

$$\Psi(s) = s\phi(\Psi(s)), \qquad 0 \leq s \leq 1\,,$$

where $s^\infty \equiv 0$. If $\mu < 1$, prove that $EY = (1-\mu)^{-1}$.

16. A particle moves at each step two units in the positive direction, with probability p, or one unit in the negative direction, with probability $q = 1-p$. If the starting position is $z > 0$, find the probability a_z that the particle will ever reach the origin. Deduce that if a fair coin is tossed repeatedly the probability that the number of heads ever exceeds twice the number of tails is $(\sqrt{5}-1)/2$.

17. Let (X_k) be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2. Show that

$$\sum_{k=1}^{n}(X_k - \bar{X})^2 = \sum_{k=1}^{n}(X_k - \mu)^2 - n(\bar{X} - \mu)^2\,,$$

where $\bar{X} = n^{-1}\sum_1^n X_k$. Prove that, if $E[(X_1-\mu)^4] < \infty$, then for every $\epsilon > 0$

$$P\left\{\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - \bar{X})^2 - \sigma^2\right| > \epsilon\right\} \to 0 \quad \text{as } n \to \infty.$$