MATHEMATICAL TRIPOS: PART IA (Lent 2002)

Probability — Example Sheet 1 (out of 4) **GRG**

Exercises

1. Let A, B, C be three events. The event *only A occurs* may be expressed as $A \cap B^c \cap C^c$. Find expressions for the events that of A, B and C,

(i) all three occur
(ii) at least one occurs
(iii) none occurs
(iv) not more than two occur.

2. In how many ways can n non-attacking rooks (i.e., no two in the same row or column) be placed on an $n \times n$ chessboard?

3. From a table of random digits, k are chosen. What are the probabilities that, for $0 \le r \le 9$,

(i) no digit exceeds r?
(ii) r is the greatest digit drawn?

For which value of r is the probability calculated in part (ii) largest?

4. Four mice are chosen (without replacement) from a litter, two of which are white. The probability that both white mice are chosen is twice the probability that neither is chosen. How many mice are there in the litter?

5. A table-tennis championship for 2^n players is organized as a knock-out tournament with n rounds, the last round being the final. Two players are chosen at random. Calculate the probabilities they meet

(i) in the first round
(ii) in the final
(iii) in any round.

[Hint: Can the same sample space be used for all three calculations?]

6. You and I play a coin-tossing game. If the coin falls heads I score one, otherwise you score one. We both start scoring from zero. Show that after $2n$ throws the probability that our scores are equal is $(2n)!/[2^n(n!)]^2$.

Use Stirling's formula to show that this probability is approximately $1/\sqrt{\pi n}$ when n is large.

7. A full deck of cards is divided into half at random. Use Stirling's formula to estimate the probability that each half contains the same number of red and black cards. [You may wish to compute the exact answers to questions **6** and **7** and compare them with your estimates.]

8. (i) If A, B, C are three events, show that

$$P(A^c \cap (B \cup C)) = P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C).$$

(ii) How many of the numbers $1, 2, \dots, 500$ are not divisible by 7 but are divisible by 3 or 5?

9. A committee of size r is chosen at random from a set of n people. Calculate the probability that m given people will all be on the committee (a) directly, and (b) using the inclusion–exclusion formula. Deduce that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

10. Examination candidates are graded into four classes known conventionally as I, II–1, II–2 and III, with probabilities 1/8, 2/8, 3/8 and 2/8 respectively. A candidate who misreads the rubric — a common event with probability 2/3 — generally does worse, his or her probabilities being 1/10, 2/10, 4/10 and 3/10. What is the probability:

(i) that a candidate who reads the rubric correctly is placed in the class II–1?
(ii) that a candidate who is placed in the class II–1 has read the rubric correctly?

11. Parliament contains a proportion p of Labour members, who are incapable of changing their minds about anything, and a proportion $1-p$ of Conservative members who change their minds completely at random (with probability r) between successive votes on the same issue. A randomly chosen member is noticed to have voted twice in succession in the same way. What is the probability that this member will vote in the same way next time?

Problems

12. There are n people gathered in a room.

(i) What is the probability that 2 (at least) have the same birthday? Calculate the probability for $n = 22, 23$.

(ii) What is the probability that at least one has the same birthday as you? What value of n makes this close to $1/2$?

13. Let f_n be the number of ways of tossing a coin n times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2, \quad f_0 = 1, \quad f_1 = 2.$$

14. If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

15. A fair coin is tossed until either the sequence HHH occurs, in which case A wins, or the sequence THH occurs, when B wins. Calculate the probability B wins.

16. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets more heads than John? Answer the same question if Mary tosses three coins and John tosses two. Make a conjecture for the probability when Mary tosses $n+1$ and John tosses n. Can you prove your conjecture?

17. The Pólya urn model for contagion is as follows. We start with an urn which contains one white ball and one black ball. At each second we choose a ball at random from the urn and replace it together with one more ball of the same colour. Is there a tendency to have a large fraction of balls of the same colour in the long run? [Computer simulations are interesting.] Calculate the probability that when n balls are in the urn, i of them are white.

18. A total of n male psychologists remembered to attend a meeting about absentmindedness. After the meeting, none could recognize his own coat so they took coats at random. Furthermore, each was liable, with probability p and independently of the others, to lose the coat on the way home. Assuming, optimistically, that all arrived home, find the probability that none had his own coat with him, and deduce that it is approximately $e^{-(1-p)}$.

19. A fair die is thrown n times. Show that the probability that there are an even number of sixes is $\frac{1}{2}(1 + (\frac{2}{3})^n)$. For the purpose of this question, 0 is an even number.

20. You throw $6n$ dice at random. Show that the probability that each number appears exactly n times is

$$\frac{(6n)!}{(n!)^6} \left(\frac{1}{6}\right)^{6n}.$$

Use Stirling's formula ($n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}$) to show that this is approximately $cn^{-5/2}$ for some constant c to be found.