MULTIDIMENSIONAL LATTICES AND THEIR PARTITION FUNCTIONS

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1. Introduction

IT is well known that limit theorems may be established for certain functions on expanding segments of regular crystalline lattices (see, for example, Ruelle ([**13**], p. 20). Such theorems are often proved for classes of functions on a particular lattice and there has been little effort to set these results in the more general context of an arbitrary multidimensional lattice. Indeed there is doubt about how such an entity should be defined. In this paper I follow a definition which is phrased in terms of the automorphism group of a graph and which was communicated to me by N. Biggs. Graphs which satisfy certain conditions to be described in § 2 have many of the properties of the commonly studied lattices (although there are other attractive conjectures which I have failed to establish). The type of interaction model which I study is a general form of that of Biggs [**3**] and is a model for interaction between the states of complete subgraphs of the lattice. When suitably normalized the ensuing partition functions of finite regions of the lattice converge as the regions expand to fill the space.

Exactly similar results hold for the random partition functions of subgraphs of lattices formed by deleting vertices independently at random; in this paper all theorems are proved in this context. The convergence here takes place in L^1, but it can be shown that the convergence is almost everywhere so long as the interaction between any eligible vertex set is nonzero. The classical interpretation of this last condition is that on any finite region each state has positive probability of occurring. The main tool is the ergodic theorem for multidimensional subadditive processes, which is the obvious extension of the work of Kingman [**9**], Grimmett [**8**] and Smythe [**15**]. This method provides sequences of upper and lower bounds for the limit function which may be used to establish certain properties of the limit. For example, in many situations the bounds are close enough to show that the convergence is uniform for some parameter, and it may be deduced that the limit function is a continuous function of that parameter.

Particular examples of interaction models include the nonisotropic Ising model and the Ashkin–Teller–Potts model; there are others with more complex interactions than between pairs of vertices only. In graph theory

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the results may be applied to the chromatic polynomial and the rank polynomial of a random lattice.

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2. Random multidimensional lattices

An *automorphism* π of a graph G is a permutation of its vertices such that u and v are adjacent vertices of G if and only if $\pi(u)$ and $\pi(v)$ are adjacent. An automorphism is said to have a *fixed point* if it maps some vertex onto itself. The collection of all automorphisms of a graph forms a group (see Biggs [**1**], p. 101). We say that a subgroup of the group of automorphisms acts without fixed points if the only member of the subgroup which does not have this property is the identity mapping. It is easy to see that a subgroup of automorphisms which acts without fixed points has the property that there is at most one automorphism which maps any vertex onto any other. For suppose that the vertices u and v are related by $\pi(u)=\rho(u)=v$ for some automorphisms π and ρ. Then $\pi^{-1}\rho(u)=u$ and so $\pi=\rho$. An *orbit* of an automorphism group $\mathcal{A}$ of G is a subset of the set of vertices of G of the form $\{\alpha(v)\colon\ \alpha\in\mathcal{A}\}$ for some vertex v.

A *d-dimensional lattice* $\mathcal{L}$ is a locally finite infinite graph which admits as a group of automorphisms the product of d infinite cyclic groups which act like $\mathbf{Z}^d$ and whose cyclic subgroups are generated by automorphisms which have no fixed points; we require further that the set of vertices of $\mathcal{L}$ has only finitely many orbits under this group of automorphisms.

This definition, which has been suggested by N. L. Biggs (personal communication), excludes many tree-like structures which are not bound together strongly by the existence of large circuits and strong regularity conditions. The generating members of the defining automorphism group of a multidimensional lattice will not in general be unique. Indeed the group itself may not be unique since any subgroup with the same properties will be sufficient to specify the lattice regularity. For example, the plane square lattice admits as a group of automorphisms the product of the two infinite cyclic groups generated by the two shift operators which translate the lattice one unit in the two axial directions. Alternatively, we might choose as generating members the shifts which translate the lattice two units in the axial directions. These automorphisms generate a group with four orbits, which contrasts with the previous single one. The original "fundamental" automorphism group might be retrieved had we chosen instead the two shifts corresponding to translation of one unit in a given axial direction and of $\sqrt{2}$ units in a diagonal direction. Thus the number of orbits of an automorphism group depends upon the generating elements rather than upon the dimensionality d alone.

All the usual crystalline lattices, such as quadratic, triangular, body-centred and face-centred lattices, are multidimensional lattices in this sense. Other examples are provided by decorating any common lattice with regularly distributed edges, or by constructing graphic networks by the regular joining of copies of any given finite graph with edges. This is illustrated by Lemma 2 below.

Multidimensional lattices have certain basic properties; the following lemmas demonstrate a few of these. I shall suppose that the defining automorphism group $\mathscr{A}$ of $\mathscr{L}$ is specified.

LEMMA 1. *Let d_{ij} be the number of vertices in the* jth *orbit which are adjacent to a given vertex in the* ith *orbit. Then $d_{ij} = d_{ji}$.*

LEMMA 2. *There exists a connected subgraph K of $\mathscr{L}$ which contains exactly one vertex from each orbit. If $\alpha, \beta \in \mathscr{A}$ then αK and βK are disjoint unless $\alpha = \beta$; hence*

$$\mathscr{L} = \bigcup_{\alpha \in \mathscr{A}} \alpha K.$$

(If G and H are subgraphs of $\mathscr{L}$ then $G \cup H$ is the subgraph of $\mathscr{L}$ whose vertices are those of G or H and whose edges are those inherited from $\mathscr{L}$).

Any subgraph K of $\mathscr{L}$ with exactly one vertex from each orbit of $\mathscr{L}$ under $\mathscr{A}$ is called a *kernel*. Kernels need not be connected. It will not be true in general that the kernel K is joined to αK (that is, some vertex of K is adjacent to some vertex of αK) whenever α is a generating element of $\mathscr{A}$. However the following holds.

LEMMA 3. *There exist automorphisms $\beta_1, \beta_2, \ldots, \beta_d$ in $\mathscr{A}$ such that K, as chosen in Lemma 2, is joined to $\beta_i K$ for $i = 1, 2, \ldots, d$.*

Note that the group $\mathscr{B}$ generated by the β's may only be a subgroup of $\mathscr{A}$, and not the whole of $\mathscr{A}$. In this case $\mathscr{B}$ will have more orbits than $\mathscr{A}$, and it is clear from the proof of Lemma 2 that K may be extended to a larger connected subgraph K' of $\mathscr{L}$ containing K such that K' contains exactly one member of each orbit of the vertices of $\mathscr{L}$ under $\mathscr{B}$.

The *covering graph* of a graph G is obtained by replacing each edge of G by a vertex and joining a pair of these vertices if and only if the corresponding two edges of G have an endpoint in common.

LEMMA 4. *The covering lattice $\mathscr{L}^c$ of a d-dimensional lattice $\mathscr{L}$ is also d-dimensional.*

I shall prove a limit theorem for certain random processes associated with $\mathscr{L}$ which requires that the interaction effects at the edge of a finite subsection of $\mathscr{L}$ are negligible when compared with internal effects. The precise lattice result is provided by the next lemma, whose notation requires explanation. Henceforth I shall suppose that $\sigma_1, \sigma_2, \ldots, \sigma_d$ are independent elements of $\mathscr{A}$ which generate $\mathscr{A}$. If $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ is an ordered set of integers, then I define $G(\mathbf{n})$ to be the subgraph of $\mathscr{L}$ given by

$$G(\mathbf{n}) = \bigcup_{-\mathbf{n}\leq \mathbf{i}\leq \mathbf{n}} \boldsymbol{\sigma}^{\mathbf{i}} K$$

where K satisfies the condition of Lemma 2, $\boldsymbol{\sigma}^{\mathbf{i}}$ is shorthand for the automorphism $\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_d^{i_d}$ where $\mathbf{i} = (i_1, \ldots, i_d)$, $\mathbf{m} \leq \mathbf{n}$ if and only if $m_j \leq n_j$ for each $j = 1, 2, \ldots, d$ and the union is over all vectors $\mathbf{i}$ with integer entries such that $-\mathbf{n} \leq \mathbf{i} \leq \mathbf{n}$. Thus $G(\mathbf{n})$ is the union of $|\mathbf{n}|$ copies of K, where $|\mathbf{n}|$ is defined to be $\prod_{j=1}^{d} (2n_j + 1)$. If $\mathscr{A}$ has k orbits, then K has k vertices and $G(\mathbf{n})$ has $k\,|\mathbf{n}|$ vertices.

LEMMA 5. *The number $e(\mathbf{n})$ of edges of $\mathscr{L}$ with exactly one endpoint in $G(\mathbf{n})$ satisfies*

$$e(\mathbf{n}) = o(|\mathbf{n}|) \quad \text{as} \quad \mathbf{n} \to \infty$$

where the limit is as $n_i \to \infty$ for each $i = 1, 2, \ldots, d$.

The following are two attractive conjectures for which I have been unable to find either proofs or counterexamples.

CONJECTURE 1. *A d-dimensional lattice is not $(d+1)$-dimensional.*

CONJECTURE 2. *Let $\mathscr{A}$ and $\mathscr{B}$ each be products of d infinite cyclic groups of automorphisms of the d-dimensional lattice $\mathscr{L}$ which both act like $\mathbf{Z}^d$, have no fixed points and have only finitely many orbits. Then there exists a third automorphism group $\mathscr{C}$ with the same properties such that $\mathscr{A}$ and $\mathscr{B}$ are subgroups of $\mathscr{C}$.*

The first conjecture is a statement of the uniqueness of dimensionality. Clearly dimensionality of $\mathscr{L}$ may be defined uniquely as the maximum value of d such that $\mathscr{L}$ is d-dimensional; the conjecture is that this can occur for only one value of d. The second conjecture asserts that there is a 'finest' automorphism group which may be used to describe the structure of the lattice. It is clear that Conjecture 1 holds if Conjecture 2 is true.

A lattice has been defined alternatively (see, for example, Rogers [12], p. 48), as all points of the form

$$\mathbf{a} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_d\mathbf{a}_d$$

where $\alpha_1, \alpha_2, \ldots, \alpha_d$ are arbitrary integers and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ are d

specified linearly independent vectors in d-dimensional space. Such a set together with a suitably imposed regular structure of edges is certainly a multidimensional lattice in the sense of this paper. This approach is concerned with the spatial configurations of points rather than their graphical properties. In a sense, any multidimensional lattice $\mathscr{L}$ may be embedded in the d-dimensional cartesian system by placing the vertices of the kernel K at suitable integer points and then copying the action of a typical automorphism $\boldsymbol{\sigma}^{\mathbf{i}}$ by the displacement of K by some fixed multiple of i_k units in the kth axial direction, for each $k = 1, 2, \ldots, d$.

I shall suppose that the vertices of the d-dimensional lattice $\mathscr{L}$ are coloured in the following random manner. Suppose that each vertex is coloured *black* with probability $p(0 \leqslant p \leqslant 1)$ independently of the colours of all other vertices; an uncoloured vertex remains *white*. In some physical models authors sometimes consider vertices to be occupied by a particle at random or subject to a spin in a certain random direction. These are special cases of the colouring process. An edge is coloured *black* if it joins a pair of black vertices, or *white* if it joins white vertices; otherwise it remains *grey*. A realization of this colouring process is represented by the subset of vertices of $\mathscr{L}$ which are coloured black.This process induces the obvious probability space $(\Omega, \mathscr{F}, P)$, where Ω is the set of all subsets of the vertex set of $\mathscr{L}$, $\mathscr{F}$ is the smallest σ-field containing the finite dimensional cyclinders $\{A \in \Omega; A \cap B = C\}$ for some finite subsets $C \subseteq B$ of Ω, and P is the induced measure on $(\Omega, \mathscr{F})$. Thus if I talk of a set A of vertices of $\mathscr{L}$ as the realization of the colouring process then I mean that the members of A are black and all others are white. This random model underlies the theory of 'site percolation' (see the review of percolation theory by Shante & Kirkpatrick [**14**]). In the theory of 'bond percolation' it is the edges of $\mathscr{L}$ rather than the vertices which are coloured black or white independently at random, but use of the covering lattice $\mathscr{L}^c$ of $\mathscr{L}$ shows that every bond percolation process on $\mathscr{L}$ can be formulated in terms of site percolation on another regular lattice (see Fisher [**6**]).

I intend to study certain types of random variables on $(\Omega, \mathscr{F}, P)$. In some applications the object of interest may be deterministic rather than random in that it may be associated with the whole lattice rather than with some random subgraph. This situation may be retrieved by setting $p = 1$.

A realization of the random colouring process is a subgraph ω of $\mathscr{L}$; I shall identify ω with its set of vertices. This induces a random subgraph $G(\mathbf{n}; \omega)$ of $G(\mathbf{n})$ obtained by deleting all vertices and edges from ω which are not in $G(\mathbf{n})$. In the next section I shall consider certain functions on $\{G(\mathbf{n}; \omega)\colon \omega \in \Omega\}$, and may sometimes write $G(\mathbf{n})$ for the quantity $G(\mathbf{n}; \omega)$ for clarity of notation.

Proof of Lemma 1. Suppose that vertex u in the ith orbit is joined to vertices $v_1, v_2, \ldots, v_m$ in the jth orbit. Then $v_i = \alpha_i(v_1)$ for $i = 2, 3, \ldots, m$ for some $\alpha_i \in \mathscr{A}$ $(i = 2, 3, \ldots, m)$, and thus v_1 is joined to $u_1, u_2, \ldots, u_m$ where $u_i = \alpha_i^{-1}(v_i)$ for $i = 2, 3, \ldots, m$; so $d_{ji} \geqslant m = d_{ij}$. The reverse inequality holds also and so $d_{ij} = d_{ji}$.

Proof of Lemma 2. Let V_i be the ith orbit of the set of vertices of $\mathscr{L}$ under $\mathscr{A}$. Let the graph G be defined by vertices $V_1, V_2, \ldots, V_k$, where k is the number of orbits, and with exactly d_{ij} edges between the pair V_i and V_j. G is a connected graph because $\mathscr{L}$ is connected; thus G has a spanning tree T. Pick any vertex v_1 from some endpoint, say V_1', of T. Vertex v_1 is joined to some vertex $v_2 \in V_2'$ where V_1' is joined to V_2' in T; for, some vertex $v_1' = \pi(v_1)$ in V_1' is joined to some v_2' in V_2', and so v_1 is joined to $v_2 = \pi^{-1}(v_2')$. Continue in this way to find V_3' joined to V_2' in T. Then v_2 is joined to some $v_3 \in V_3'$ by the previous argument and this process may be continued until all the vertices of T are exhausted. We have then constructed a connected subgraph K with vertices $\{v_1, v_2, \ldots, v_k\}$ with exactly one vertex in each orbit.

If $\alpha, \beta \in \mathscr{A}$ and $\alpha \neq \beta$, then αK and βK are disjoint because if they have some vertex w in common then $w = \alpha(u) = \beta(v)$ for some $u, v \in K$. But u and v are in the same orbit of $\mathscr{L}$ under $\mathscr{A}$ and thus $u = v$, giving $\alpha = \beta$ by the absence of fixed points. Every vertex in $\mathscr{L}$ is in some orbit of $\mathscr{L}$ under $\mathscr{A}$ and so is in some αK as α ranges over $\mathscr{A}$. Hence

$$\mathscr{L} = \bigcup_{\alpha \in \mathscr{A}} \alpha K.$$

How many nonisomorphic kernels K can be chosen to satisfy the conditions of the lemma? Clearly this number does not exceed the number of spanning trees of the multigraph G; it will be strictly less than the number of spanning trees if some kernel contains a circuit.

Proof of Lemma 3. Suppose the kernel K is chosen according to Lemma 2 and has vertices $\{a, b, c, \ldots, k\}$. Some vertex of K is joined to a vertex not in K and in a different orbit. That is, we may assume that a is joined to $\beta_1(b)$ for some $\beta_1 \in \mathscr{A}$ which is not the identity. Then K is joined to $\beta_1 K$. Let

$$G_1 = \bigcup_{i=-\infty}^{\infty} \beta_1^i K.$$

Clearly some point in G_1 is joined to some point not in G_1 and in some other orbit than the first. Therefore some vertex in K, say c, is joined to $\beta_2(d)$ for some $\beta_2 \in \mathscr{A}$, not the identity mapping. Then K is joined to $\beta_2 K$, where β_2 is independent of β_1. Now let

$$G_2 = \bigcup_{i,j} \beta_1^i \beta_2^j K$$

and proceed as before. We may continue to generate a sequence $\beta_1, \beta_2, \ldots$ of independent automorphisms which will terminate when the subgroup of $\mathcal{A}$ which they generate is the product of d cyclic groups generated by themselves. The invariance of rank of free abelian groups guarantees that we may find exactly d such automorphisms. Note that K may not be a kernel for the subgroup $\mathcal{B}$ of $\mathcal{A}$ generated by the β's. But it is clear from the proof of Lemma 2 that K may be extended to a larger connected subgraph K' which may serve as a kernel for $\mathcal{B}$.

Proof of Lemma 4. Let (u, v) be an edge of $\mathcal{L}$. The automorphism α of $\mathcal{L}$ induces the obvious automorphism α' of $\mathcal{L}^c$ defined by $\alpha'((u, v)) = (\alpha(u), \alpha(v))$, and it follows easily that $\mathcal{L}^c$ possesses a free abelian group of rank d as an automorphism group which acts with finitely many orbits and without fixed points.

Proof of Lemma 5. Let $\mathbf{m}$ be any vector with positive integral entries such that no vertex K is joined to any vertex not in $G(\mathbf{m})$. Such an $\mathbf{m}$ certainly exists because there are only finitely many vertices of $\mathcal{L}$ joined to K and all these may be included in some $G(\mathbf{m})$ by picking the entries of $\mathbf{m}$ large enough. Then if $\mathbf{n}>\mathbf{m}$ no vertex in $G(\mathbf{n}-\mathbf{m})$ $(\mathbf{n}-\mathbf{m}=(n_1-m_1, \ldots, n_d-m_d))$ is joined to any vertex not in $G(\mathbf{n})$. Thus each edge contributing to $e(\mathbf{n})$ originates from a vertex in $G(\mathbf{n})-G(\mathbf{n}-\mathbf{m})$, and so

$$e(\mathbf{n}) \leq \delta k(|\mathbf{n}|-|\mathbf{n}-\mathbf{m}|)$$

where δ is the maximum vertex degree of $\mathcal{L}$ and k is the number of orbits. But

$$\frac{|\mathbf{n}-\mathbf{m}|}{|\mathbf{n}|}=\prod_{j=1}^{d}\left(1-\frac{2m_j}{(2n_j+1)}\right)\to 0 \quad \text{as} \quad \mathbf{n}\to\infty$$

and it follows that

$$\frac{e(\mathbf{n})}{|\mathbf{n}|}\to 0$$

as required.

3. Interaction models and partition functions

The interaction model which I describe here is a general form of that defined by Biggs [3] [4] who considers pair interaction. It is a general model for nearest neighbour interaction between the vertices of a graph. A limit theorem will be necessary to show the existence of the so-called 'partition function' for certain infinite graphs; this limit will only exist if the graph is sufficiently regular, and in this paper I shall suppose later that it is actually a multidimensional lattice.

Suppose that Σ is a finite set of *states*; each vertex of a finite graph G may be in any one of the states Σ. A *state* of the graph G is an allocation of states, one to each vertex of G, and $\Sigma(G)$ $(=\Sigma^v$, where v is the number of vertices of G) denotes the set of all states of G, or the *state space* of G. A *clique* of G is a complete subgraph of G. The letters s and C will be used to represent a typical state and a typical clique of G respectively. The symbols sC will represent the state of the subgraph C which is induced by s. That is, sC is the state of C obtained by allocating the same state to each vertex of C as is allocated by the state s of G; it is called a *clique state*. Let $\mathscr{C}S(G)$ denote the collection of all clique states of G. An *interaction function* on G is a function ϕ: $\mathscr{C}S(G) \to \mathbb{R}$. For any state s in $\Sigma(G)$ the *weight* of s is defined to be

$$w(s) = \prod_{C \subseteq G} \phi(sC)$$

where the product is over all cliques C contained in G. The *partition function* of G is

$$Z(G) = \sum_{s \in \Sigma(G)} w(s),$$

where the sum is over all states s of G. Classically, if ϕ is a non-negative function then the weight of s is a measure of the probability that the state occurs in the equilibrium distribution of some random process; thus the exact probability that s occurs is $w(s)/Z(G)$. Such probability measures on the set of states of a finite graph are known as Gibbs ensembles if $\phi > 0$, and they may be characterized as those measures for which the distribution of the random state of any given vertex depends only upon the states of its neighbours. For a more precise statement of this property see Grimmett [**7**]. Preston [**11**] investigates similar measures on countably infinite graphs.

The partition function of a graph contains information about its macroscopic state, and relates this to the underlying microscopic parameters. The interesting cases are mostly associated with infinite graphs, for which care is needed to ensure the existence of such a function. See the next section.

This definition of an interaction function differs in two respects from that of Biggs [**3**], who considers only cliques C of size two (edges) and specifies that their contribution $\phi(sC)$ towards the weight depends only upon the states of the endpoints of C and not upon the position of C in the graph G.

Interaction functions are commonly studied in theoretical physics, where $\phi(sC)$ measures the strength of interaction of the clique C when its vertices are in the states allocated by s. We say that there is pair interaction only if $\phi(sC)=1$ whenever C contains more than two vertices. Particular cases of pair partition functions of a graph G include the chromatic polynomial of G and the classical partition functions of the Ising and Ashkin–Teller–Potts models on G.

(a) *Chromatic polynomial.* The chromatic polynomial $P(\lambda)$ of G is the number of ways of colouring the vertices of G with λ colours in such a way that no pair of adjacent vertices have the same colour. Suppose that Σ is the set of available colours and that ϕ is given by $\phi(sC)=0$ if C has two vertices which are allocated the same states by s, and $\phi(sC)=1$ otherwise. It is easy to see that $P(\lambda)$ is the partition function of the corresponding interaction model.

(b) *Ising model.* The Ising model is the interaction model given by $\Sigma=\{-1,1\}$ and

$$\phi(sC)=\begin{cases}\exp(H\sigma) & \text{if } |C|=1\\ \exp(J(\sigma_1\sigma_2-1)) & \text{if } |C|=2\\ 1 & \text{if } |C|>2\end{cases}$$

where σ, σ_1 and σ_2 are the states allocated to the vertices of C in each case respectively. The partition function is then given by

$$Z(G)=\sum_{(\sigma)}\exp\Big(H\sum_v\sigma+J\sum_e(\sigma_1\sigma_2-1)\Big)$$

where the first sum is over all states of G, $\sum_v$ sums over all vertices, $\sum_e$ sums over all edges and σ, σ_1 and σ_2 are the states of the appropriate vertices. H and J are constants, measuring the external field and the strength of interaction respectively.

(c) *Ashkin–Teller–Potts model.* Here

$$\Sigma=\{-S,1-S,\ldots,-1,0,1,\ldots,S-1,S\}$$

and ϕ is given by

$$\phi(sC)=\begin{cases}\exp(H\sigma) & \text{if } |C|=1\\ \exp(-2J\varepsilon(\sigma_1,\sigma_2)) & \text{if } |C|=2\\ 1 & \text{if } |C|>2\end{cases}$$

where σ, σ_1 and σ_2 are as before, and

$$\varepsilon(\sigma_1,\sigma_2)=\begin{cases}0 & \text{if } \sigma_1\sigma_2>0\\ 1 & \text{if } \sigma_1\sigma_2\leqslant 0.\end{cases}$$

Essam [**5**] has shown that these three partition functions are also derivable from the Whitney rank polynomial (see Biggs [**1**], p. 64) which is given by

$$W_G(x,y)=\sum_{E'\subseteq E(G)}x^{v-k}y^{e-v+k}$$

where v is the number of vertices of G, e and k are the numbers of edges and components of the subgraph of G with edge set E', and the summation is over all subsets E' of the edge set of G.

However, the rank polynomial may be used to model only pair interactions. It is not itself a partition function although it is entirely deducible from a knowledge of the partition functions of certain interaction models. For, up to a factor which may be dealt with separately, $W_G(x, y)$ is the same as the partition function of the Ashkin–Teller–Potts model with $\lambda = y/x$ states; this is a polynomial in λ of finite order and therefore knowledge of it for all positive odd integral values of λ specifies W_G exactly. I shall be interested in limit theorems for partition functions as the underlying graph expands to fill the space available; the limit function will not in general be a finite polynomial, and thus knowledge of the limit for integral values of λ does not imply knowledge of the corresponding limit function of the rank polynomials.

The rank polynomial of a random graph contains a lot of information about its structure. In particular the rank polynomials of random lattices are significant quantities in percolation theory (see, for example, Temperley and Lieb [**16**]).

A feature common to all three examples is that ϕ takes the same value on any pair of cliques of the same size and in the same state. Clearly, more complex interactions, such as in the nonisotropic Ising model in which the strength of interaction along an edge depends upon the direction of that edge, may be modelled by a more carefully defined interaction function ϕ which distinguishes between the differing spatial characteristics of the cliques.

In each of the three previous examples the interaction function ϕ takes non-negative values only; I shall assume henceforth that this holds always.

4. The limit theorem for partition functions of random lattices

Suppose that $\mathcal{L}$ is a multidimensional lattice and Σ a finite set of states. I shall suppose that $\mathcal{L}$ is random in the sense described earlier: each vertex is coloured black with probability $p (0 \leq p \leq 1)$; otherwise it remains white. Ω denotes the set of all subsets of the vertex set of $\mathcal{L}$, and $\omega \in \Omega$ denotes a realization of the colouring process, as before. Let ϕ be an interaction function defined on the set of clique states of $\mathcal{L}$. We say that ϕ is *invariant* with respect to the subgroup $\mathcal{A}$ of automorphisms of $\mathcal{L}$ if

$$\phi(sC) = \phi(s(\alpha C))$$

for all states s, cliques C of $\mathcal{L}$ and $\alpha \in \mathcal{A}$. Henceforth I shall suppose that

$\mathscr{A}$ is the group of automorphisms used to define $\mathscr{L}$, and that ϕ is invariant with respect to $\mathscr{A}$. This condition imposes a regularity on the behaviour of ϕ. For a given multidimensional lattice $\mathscr{L}$, defining group $\mathscr{A}$ and interaction function ϕ, it may be that this last invariance condition is not satisfied; but ϕ may be invariant with respect to some subgroup $\mathscr{B}$ of $\mathscr{A}$, also being a free abelian group of rank d acting without fixed points and with finitely many orbits. In this case we may take $\mathscr{B}$ to be the defining automorphism group. Thus the automorphism group and interaction function may be seen as a pair of objects which jointly satisfy certain conditions. The main theorem deals with the behaviour of the random variables

$$Z_{\mathbf{n}}(\omega) = Z(G(\mathbf{n}; \omega))$$

as $\mathbf{n} \to \infty$. The symbols M and m will represent the maximum and minimum values, respectively, which $\phi(sC)$ attains as sC ranges over all states on all cliques in $\mathscr{L}$. They are finite because ϕ takes finitely many values only.

THEOREM 1. *If* $m > 0$, *then* $Y_{\mathbf{n}} = \log Z_{\mathbf{n}}$ *satisfies*

$$\frac{Y_{\mathbf{n}}}{|\mathbf{n}|} \to \gamma \quad \textit{a.e. and in } L^1 \textit{ as} \quad \mathbf{n} \to \infty$$

where γ *is a constant given by*

$$\gamma = \inf_{\mathbf{r}} ((E(Y_{\mathbf{r}}) + e(\mathbf{r})p^2 \log L)/|\mathbf{r}|)$$

and

$$\gamma = \sup_{\mathbf{r}} ((E(Y_{\mathbf{r}}) + e(\mathbf{r})p^2 \log l)/|\mathbf{r}|)$$

for some given L and l. The convergence is uniform on any compact subset of the set of all possible values of any parameter in terms of which ϕ *is defined.*

By setting $p = 1$ we retrieve the deterministic case, which contains the chromatic polynomial and Ising partition function as particular examples. The theorem provides sequences of upper and lower bounds for γ which are asymptotically sharp and which may be used to deduce properties of γ:

$$e(\mathbf{r})p^2 \log l \leq \gamma\, |\mathbf{r}| - E(Y_{\mathbf{r}}) \leq e(\mathbf{r})p^2 \log L, \quad \text{for all } \mathbf{r}.$$

A similar general limit theorem for partition functions is shown by Ruelle [**13**]. There is similarity between our arguments, although it is clear that his proof for classical interactions applies in a more limited context and does not indicate the crucial importance of the central feature of subadditivity in the proof which follows.

The next theorem provides a partial description of the degree to which the limit γ is independent of the structure used to establish its existence.

THEOREM 2. *The limit γ depends only upon the group $\mathcal{A}$ and not upon the choice of generating elements $\sigma_1, \sigma_2, \ldots, \sigma_d$ or kernel K. If $\mathcal{B} \subseteq \mathcal{A}$ where $\mathcal{B}$ is a free abelian group of rank d with finitely many orbits then γ is also the limit associated with $\mathcal{B}$.*

If $m = 0$ a similar L^1 limit theorem holds for the $Y_{\mathbf{n}}$ so long as $\gamma > -\infty$, and the method used to prove the theorem shows that

$$\frac{Y_{\mathbf{n}}}{|\mathbf{n}|} \to \gamma \quad \text{in} \quad L^1 \text{ as} \quad \mathbf{n} \to \infty.$$

If $\gamma = -\infty$ then it is easy to see that

$$\frac{E(Y_{\mathbf{n}})}{|\mathbf{n}|} \to -\infty \quad \text{as} \quad \mathbf{n} \to \infty$$

while $E(\max(0, Y_{\mathbf{n}}/|\mathbf{n}|))$ is uniformly bounded for all $\mathbf{n}$. Convergence a.e. cannot of course be shown by these methods unless $m > 0$.

Proof of Theorem 1. Let H and J be two finite disjoint subgraphs of a graph G such that no edge of G joins a vertex of H to a vertex of J. It is clear that $Z(H \cup J) = Z(H)Z(J)$ because there are no cliques of $H \cup J$ which intersect both H and J. Now suppose that H and J are joined by exactly e edges of G. Then

$$Z(H \cup J) = \sum_{\substack{s \in \Sigma(H) \\ t \in \Sigma(J)}} \prod_{C \subseteq H} \phi(sC) \prod_{C \subseteq J} \phi(tC) \prod_{C}{}' \phi(stC)$$

where $\prod'$ is the product over all cliques C in $H \cup J$ which intersect both H and J and st is the state of $H \cup J$ obtained by combining s on H with t on J. The first two products are over all cliques C contained in H and J respectively. Each of the e edges linking H and J is contained in at most $\binom{\delta-1}{c-2}$ cliques of size c, where δ is the maximum vertex degree of G, and thus

$$Z(H \cup J) \leq Z(H)Z(J)\left(M_2 M_3^{\binom{\delta-1}{1}} M_4^{\binom{\delta-1}{2}} \cdots M_{\delta+1}^{\binom{\delta-1}{\delta-1}}\right)^e$$

where $M_i = \max\{\phi(sC)\colon\ s \in \Sigma(G),\ C \text{ is a clique of } G \text{ with } i \text{ vertices}\}$. Setting $M = \max\{M_1, M_2, \ldots, M_{\delta+1}\}$ we get

$$Z(H \cup J) \leq Z(H)Z(J)M^{e2^{\delta-1}}, \tag{1}$$

although this bound may be improved if we know some limitation on the range of interaction of ϕ, such as that there is only pair interaction.

Writing $L = M^{2^{d-1}}$ we have that

$$Z'(H \cup J) \leq Z'(H)Z'(J) \tag{2}$$

where $Z'(H) = Z(H)L^{e(H)}$ for any subgraph H of G and $e(H)$ is the number of edges of G with exactly one endpoint in H. Exactly the same argument holds to show that

$$Z''(H \cup J) \geq Z''(H)Z''(J) \tag{3}$$

where $Z''(H) = Z(H)l^{e(H)}$ for any subgraph H of G, $e(H)$ is as before and $l = m^{2^{d-1}}$.

Let $Y'(G) = \log Z'(G)$. Then the process $\{Y'\}$ is subadditive in the sense that $Y'(H \cup J) \leq Y'(H) + Y'(J)$. Defining $Y'_{\mathbf{n}}(\omega) = \log Z'(G(\mathbf{n}; \omega))$ and using the invariance of ϕ we obtain a d-dimensional subadditive stochastic process; such processes have been studied in one and two dimensions by Kingman [**9**] [**10**], Grimmett [**8**] and Smythe [**15**], whose papers contain limit theorems for such processes. A general result of Smythe [**15**] for two-dimensional processes may be extended in the obvious way to larger dimensions to demonstrate that

$$\frac{Y'_{\mathbf{n}}}{|\mathbf{n}|} \to \xi \quad \text{in } L^1 \text{ as } \quad \mathbf{n} \to \infty \tag{4}$$

where

$$E(\xi) = \gamma = \inf_{\mathbf{r}} \left(E(Y'_{\mathbf{r}})/|\mathbf{r}|\right)$$

provided that $E\,|Y'_{\mathbf{r}}|/|\mathbf{r}| < A$ for some A and all $\mathbf{r}$. This uniform bound holds in our case for, by (1) and (3),

$$(m\,|\Sigma|)^v B_1^e \leq Z(G) \leq (M\,|\Sigma|)^v B_2^e$$

for any graph G with v vertices and e edges, and for some fixed positive constants B_1 and B_2. Thus

$$C_1 \leq \frac{Y'(G)}{v} \leq C_2$$

for fixed positive constants C_1 and C_2. In the case $m = 0$ there may be no uniform bound A, since we now only have the upper bound

$$\frac{Y'(G)}{v} \leq C_2$$

and it may be the case that

$$\inf_{\mathbf{r}} \left(E(Y'_{\mathbf{r}})/|\mathbf{r}|\right) = -\infty.$$

If this is so then we may conclude from the usual argument used in

Grimmett [8] and Smythe [15] that equation (4) holds in the sense that

$$E(Y'_{\mathbf{r}})/|\mathbf{r}| \to -\infty \quad \text{as} \quad \mathbf{r} \to \infty$$

although $E(\max(0, Y'_{\mathbf{r}}))/|\mathbf{r}| < A'$ for some A' and all $\mathbf{r}$.

Note that ξ is constant almost everywhere. For, let $v_1, v_2, \ldots$ be an ordering of the vertices in the lattice, and let $X_i(\omega)$ be the random colouring of v_i under ω. It is not hard to see that ξ is $\mathcal{T}$-measurable where $\mathcal{T}$ is the tail σ-field of the sequence $\{X_i: i \geq 1\}$ of independent random variables. But $\mathcal{T}$ contains only trivial events and it follows that $\xi = \gamma$ a.e.

In both cases $m = 0$ and $m > 0$ the general theory of subadditive processes shows that

$$P(\limsup Y'_{\mathbf{n}}/|\mathbf{n}| \leq \gamma) = 1. \tag{5}$$

To conclude the proof of L^1 convergence of $Y_{\mathbf{n}}/|\mathbf{n}|$ note that

$$Y'_{\mathbf{n}}(\omega) = Y_{\mathbf{n}}(\omega) + e(\mathbf{n}; \omega) \log L. \tag{6}$$

where $e(\mathbf{n}; \omega)$ is the number of edges of the black subgraph ω of $\mathcal{L}$ with exactly one vertex in $G(\mathbf{n})$. But

$$e(\mathbf{n}; \omega)/|\mathbf{n}| \to 0 \tag{7}$$

by Lemma 5, and so $Y'_{\mathbf{n}}/|\mathbf{n}|$ and $Y_{\mathbf{n}}/|\mathbf{n}|$ have the same L^1 limit γ given by

$$\gamma = \inf_{\mathbf{r}} (E(Y'_{\mathbf{r}})/|\mathbf{r}|)$$

$$= \inf_{\mathbf{r}} (E(Y_{\mathbf{r}}) + e(\mathbf{r})p^2 \log L)/|\mathbf{r}|.$$

It follows also from (5), (6) and (7) that

$$P(\limsup Y_{\mathbf{n}}/|\mathbf{n}| \leq \gamma) = 1. \tag{8}$$

A general result of Smythe would imply that the convergence would take place almost everywhere if the process $Y'_{\mathbf{n}}$ were "strongly subadditive" in that it satisfies a more stringent family of inequalities than the usual subadditive relations. This will not hold in general and so it is necessary to use the alternative argument provided by the previous observation that $\{Y_{\mathbf{n}}\}$ is almost superadditive as well as being almost subadditive. That is, if $m > 0$ then, by (3), $Y''(G) = \log Z''(G)$ satisfies

$$Y''(H \cup J) \geq Y''(H) + Y''(J)$$

and the same argument as before may be used to deduce the following limiting behaviour of $Y''_{\mathbf{n}} = Y''(G(\mathbf{n}; \omega))$:

$$Y''_{\mathbf{n}}/|\mathbf{n}| \to \xi' \quad \text{in } L^1 \text{ as } \mathbf{n} \to \infty$$

where

$$E(\xi') = \gamma' = \sup_{\mathbf{r}} (E(Y''_{\mathbf{r}})/|\mathbf{r}|).$$

As before $\xi' = \gamma'$ a.e.; furthermore it is easy to see that $\gamma' = \gamma$, because $Y''_{\mathbf{n}}$ and $Y'_{\mathbf{n}}$ differ by a quantity which is negligible compared with $|\mathbf{n}|$, and so $\gamma' = \lim Y''_{\mathbf{n}}/|\mathbf{n}| = \lim Y_{\mathbf{n}}/|\mathbf{n}| = \gamma$. Also

$$\liminf Y_{\mathbf{n}}/|\mathbf{n}| \geq \gamma' = \gamma \quad \text{a.e.} \tag{9}$$

Combining (8) and (9) yields

$$Y_{\mathbf{n}}/|\mathbf{n}| \to \gamma \quad \text{a.e. and in } L^1.$$

The two derivations of γ provide the sequences of bounds

$$(E(Y_{\mathbf{r}}) + e(\mathbf{r})p^2 \log l)/|\mathbf{r}| \leq \gamma \leq (\mathrm{E}(Y_{\mathbf{r}}) + e(\mathbf{r})p^2 \log L)/|\mathbf{r}|. \tag{10}$$

These bounds show that the convergence of $\gamma_{\mathbf{r}} = E(Y_{\mathbf{r}})/|\mathbf{r}|$ to γ is uniform on any compact subset of the values of any parameter in terms of which ϕ is defined. For

$$(e(\mathbf{r})p^2 \log l)/|\mathbf{r}| \leq \gamma - \gamma_{\mathbf{r}} \leq (e(\mathbf{r})p^2 \log L)/|\mathbf{r}|$$

and so the difference $\gamma - \gamma_{\mathbf{r}}$ may be made arbitrarily small for large $\mathbf{r}$ independently of the value taken by such a parameter, or indeed by the underlying probability p. Thus γ is a continuous function of p and any other parameter of which ϕ is a continuous function.

Proof of Theorem 2. I will omit some of the easier details of the proof. Suppose that $\tau_1, \tau_2, \ldots, \tau_d$ are independent members of $\mathscr{A}$ which generate $\mathscr{A}$ and which are given by $\tau_i = \boldsymbol{\sigma}^{\mathbf{a}_i}$ where $\mathbf{a}_i^T = (a_{1i}, a_{2i}, \ldots, a_{di})^T$ is the ith column of the matrix $A = (a_{ij})$. Then

$$H(\mathbf{n}) = \bigcup_{-\mathbf{n} \leq \mathbf{i} \leq \mathbf{n}} \boldsymbol{\tau}^{\mathbf{i}} K$$

is the union of all αK where $\alpha \in \mathscr{A}$ is of the form $\boldsymbol{\sigma}^{\mathbf{j}}$ for some $\mathbf{j} = \mathbf{i}A$ where $-\mathbf{n} \leq \mathbf{i} \leq \mathbf{n}$. Then $\mathbf{i} = \mathbf{j}A^{-1}$ and $H(\mathbf{n})$ is the union over all $\boldsymbol{\sigma}^{\mathbf{j}}K$ for points $\mathbf{j}$ in the region of $\mathbf{Z}^d$ defined by $-\mathbf{n} \leq \mathbf{j}A^{-1} \leq \mathbf{n}$. This is a polytope which may be decomposed into the union of graphs like $\boldsymbol{\sigma}^{\mathbf{k}}G(\mathbf{r})$ for certain values of $\mathbf{k}$ and any fixed $\mathbf{r}$ together with some remainder region at the edge of the polytope. Following the usual proof for convergence of multidimensional processes we may observe that the remainder region has a negligible effect upon the partition function of $H(\mathbf{n})$ for large $\mathbf{n}$ when compared with the internal effects. Thus the limit exists as $\mathbf{n} \to \infty$. It is now defined in terms of the $G(\mathbf{n})$ and therefore coincides with γ. To show that the limit a.e. is the same for both sets of generators we would require that $m > 0$, since otherwise we cannot even guarantee that either limit exists.

Next I show that the choice of the kernel K does not affect γ. Indeed it is not even important that K be connected. For suppose that K_1 is some other subgraph, not necessarily connected, with exactly one vertex from each orbit of $\mathscr{L}$ under $\mathscr{A}$. There exists $\mathbf{j} = (j_1, j_2, \ldots, j_d)$ such that

$$K_1 \subseteq \bigcup_{-\mathbf{j} \leq \mathbf{l} \leq \mathbf{j}} \sigma^{\mathbf{l}} K$$

since every vertex of $\mathscr{L}$ is in αK for some $\alpha \in \mathscr{A}$. It follows that

$$K \subseteq \bigcup_{-\mathbf{j} \leq \mathbf{l} \leq \mathbf{j}} \sigma^{\mathbf{l}} K_1.$$

Now

$$G_1(\mathbf{n}-\mathbf{j}) \subseteq G(\mathbf{n}) \subseteq G_1(\mathbf{n}+\mathbf{j})$$

where $G_1(\mathbf{n}) = \bigcup_{-\mathbf{n} \leq \mathbf{l} \leq \mathbf{n}} \sigma^{\mathbf{l}} K_1$ and $\mathbf{r}+\mathbf{s}$ is the vector with entries $r_i + s_i$ $(i = 1, 2, \ldots, d)$. Thus by (1)

$$Z(G_1(\mathbf{n}-\mathbf{j};\, \omega))A \leq Z(G(\mathbf{n};\, \omega)) \leq Z(G_1(\mathbf{n}+\mathbf{j};\, \omega))B \tag{11}$$

where $A = Z_1 L^{-2e'}$, $B = Z_2 L^{-2e''}$, e' and e'' are the numbers of edges of ω with exactly one endpoint in $G_1(\mathbf{n}-\mathbf{j};\, \omega)$ and $G(\mathbf{n};\, \omega)$ respectively, and $Z_1 = Z(G(\mathbf{n};\, \omega) - G_1(\mathbf{n}-\mathbf{j};\, \omega))$ and $Z_2 = Z(G_1(\mathbf{n}+\mathbf{j};\, \omega) - G(\mathbf{n};\, \omega))$. But, using (1) and (3), we see that

$$Z_1^{|\mathbf{n}|^{-1}}, Z_2^{|\mathbf{n}|^{-1}} \to 1 \quad \text{a.e. as} \quad \mathbf{n} \to \infty$$

because

$$\frac{|\mathbf{n}| - |\mathbf{n}-\mathbf{j}|}{|\mathbf{n}|}, \frac{|\mathbf{n}+\mathbf{j}| - |\mathbf{n}|}{|\mathbf{n}|} \to 0 \quad \text{as} \quad \mathbf{n} \to \infty.$$

Using Lemma 5 we conclude that

$$A^{|\mathbf{n}|^{-1}}, B^{|\mathbf{n}|^{-1}} \to 1 \quad \text{a.e. as} \quad \mathbf{n} \to \infty$$

and thus $\varliminf_{\mathbf{n}\to\infty} (\log Z(G_1(\mathbf{n};\, \omega))/|\mathbf{n}|) = \varliminf_{\mathbf{n}\to\infty} (\log Z(G(\mathbf{n};\, \omega))/|\mathbf{n}|)$ a.e. from (11).

Finally suppose that $\mathscr{B} \subseteq \mathscr{A}$ is a subgroup of the group $\mathscr{A}$ and is also a free abelian group of rank d. Clearly ϕ is invariant with respect to $\mathscr{B}$. Let K be a kernel for $\mathscr{B}$ and suppose that $\mathscr{A}$ is generated by $\sigma_1, \sigma_2, \ldots, \delta_d$. Then each vertex of K is in a different orbit of $\mathscr{L}$ under $\mathscr{B}$, and so K may be imbedded in some larger subgraph of the form $K' = \bigcup_{\alpha \in \mathscr{D}} \alpha K$, where $\mathscr{D}$ is some finite subset of $\mathscr{A}$, such that K' serves as a kernel for $\mathscr{B}$. There exists a kernel for $\mathscr{B}$ of this form because, for any α, the orbit containing some given vertex of αK under $\mathscr{B}$ intersects K if and only if the orbits containing each vertex of αK under $\mathscr{B}$ intersect K. Now we use similar arguments to those used already in this proof to show that the limit for $\mathscr{B}$ is the same as the limit for $\mathscr{A}$.

Finally note that if, for any state s and clique C, $\phi(sC)$ depends only upon the size of C and the states of its vertices then the limit γ depends only upon the graphical structure of $\mathscr{L}$ and not upon its defining group of automorphisms, so long as Conjecture 2 is true. This holds since the truth of the conjecture implies that for any defining groups $\mathscr{A}$ and $\mathscr{B}$ of automorphisms, there exists a third, $\mathscr{C}$, which contains both $\mathscr{A}$ and $\mathscr{B}$. And by the last part of the theorem the limits for $\mathscr{A}$ and $\mathscr{B}$ both coincide with the limit for $\mathscr{C}$.

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