ON COUNTING POLYGONS IN A CRYSTAL

GEOFFREY R. GRIMMETT

Abstract. How many n-step polygons exist that contain a given vertex of an infinite quasi-transitive graph G? The exponential growth rate of such polygons is identified as the connective constant when G has sub-exponential growth and possesses a so-called square graph height function. The last amounts to the requirement that G has a $\mathbb{Z}^2$ action of automorphisms. The main theorem extends a result of Hammersley for the hypercubic lattice (Proc. Cambridge Philos. Soc. 57 (1961) 516–523), and responds to his challenge to prove such a result for more general "crystals".

1. Introduction

A *self-avoiding walk* (SAW) on a graph G is a path that visits no vertex more than once. The study of SAWs was initiated in the chemical theory of polymerisation (see [25] and the book [6] of Flory). In their visionary paper [15], Hammersley and Morton studied *inter alia* the number of n-step SAWs on a lattice G. They used subadditivity to prove the existence of the exponential growth rate, that is, the limit

$$\mu(G) := \lim_{n\to\infty} c_n^{1/n}, \tag{1.1}$$

where c_n is the number of n-step SAWs on G starting at a given vertex. This investigation was continued by Hammersley alone in [13]. The constant $\mu(G)$ was termed the *connective constant* of G in [3], where (1.1) is stated without proof.

In 1961, Hammersley [14] extended the theory from counts of SAWs to counts of (self-avoiding) polygons. He showed that, in the case of the hypercubic lattice $\mathbb{Z}^d$, the exponential growth rates of the numbers of n-step SAWs and polygons are equal. The purpose of the current note is to provide a response to Hammersley's challenge to determine "what properties a crystal must possess" in order that this be the case, while noting that our answer is sufficient but probably not necessary.

In a further work [16], Hammersley and Welsh introduced a technique that has proved very useful in studying SAWs and polygons, namely the combinatorics of so-called 'bridges'. In broad terms made specific in Section 2, a bridge is a SAW with

Date: 9 December 2025.
2010 *Mathematics Subject Classification.* 05C30, 05C38, 60K35, 82B20.
Key words and phrases. self-avoiding polygon, self-avoiding walk, connective constant.

extremal endpoints. The key property of bridges is that one may extend a bridge b by appending a second bridge b' to the final vertex of b. It is proved in [16] that, for the hypercubic lattice $\mathbb{Z}^d$, the exponential growth rate of the number of n-step bridges equals the connective constant $\mu(\mathbb{Z}^d)$.

In the current paper, we study polygons in quasi-transitive graphs G satisfying certain conditions, thereby extending results of [14] beyond the hypercubic lattices $\mathbb{Z}^d$. The main result of this paper is Theorem 3.2, which we summarise as follows.

Theorem 1.1 (Equality of growth rates). *Let $G \in \mathcal{G}$ have a square graph height function and sub-exponential growth, and let p_n be the number of n-step polygons that include the root of G. Then $\pi(G) := \limsup_{n\to\infty} p_n^{1/n}$ satisfies $\pi(G) = \mu(G)$.*

The set $\mathcal{G}$ and the term 'square graph height function' are explained in Section 2. Some illustrations of the terms and conclusion of this theorem are provided in Examples (3.3)–(3.6). A discussion of the missing liminf is found after Theorem 3.2.

The challenge in the current work lies in doing without some of the symmetries of $\mathbb{Z}^d$ that underly earlier work. This may be seen as a continuation of the work of Grimmett and Li directed at understanding the properties of connective constants of general quasi-transitive graphs (see [11] for a review).

Theorem 1.1 provides a sufficient condition on the graph G for the equality $\pi(G) = \mu(G)$ to hold. It is believed that the strict inequality $\pi(G) < \mu(G)$ holds for non-amenable graphs, and some comments on this inequality and its connection to the property of ballisticity are included at the end of Section 3.

The reader is referred to [2, 7, 23] and [18, Chap. 7] for general accounts of the combinatorics of SAWs and their cousins, and to [11] for SAWs on quasi-transitive graphs.

This paper is structured as follows. Section 2 is devoted to background terminology and properties for graphs and their height functions. The principal Theorem 3.2 follows in Section 3, together with some examples and a discussion of the relationship to the ballisticity of SAWs. The proof is found in Section 4.

2. Preliminaries

Throughout this paper, $G = (V, E)$ will denote an infinite, connected, locally finite, quasi-transitive graph with root denoted $\mathbf{0}$. For simplicity, we assume G has neither parallel edges nor loops, and we write $\mathcal{G}$ for the set of all such rooted graphs. If u and v are neighbours in G (written $u \sim v$), we write $\langle u, v\rangle$ for the edge joining them. The set of neighbours of v is denoted ∂v. The graph-distance $d(u, v)$ is the number of edges in the shortest path from u to v.

The automorphism group of $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to *act transitively* on G (or on its vertex-set V) if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma(v) = w$. The subgroup Γ is said to *act quasi-transitively* if there is a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma(v) = w$. The graph G is called *transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). The orbit of a vertex v under Γ is denoted Γv. The number of orbits of V under Γ is written as $M(\Gamma) = |V/\Gamma|$.

A *cycle* of G is a sequence $(v_0, v_1, \dots, v_m)$ with $m \geq 3$ such that $v_i \sim v_{i+1}$ for $0 \leq i < m$, $v_m = v_0$, and $v_0, v_1, \dots, v_{m-1}$ are distinct vertices. The length of a cycle is the number of edges traversed. There is some indecision in the literature concerning the terms cycle, circuit, polygon, and we shall define the last in the next paragraph.

A *self-avoiding walk* (SAW) on G is a path starting at $\mathbf{0}$ that visits no vertex more than once. A *polygon* is a cycle of G containing the root $\mathbf{0}$; that is, a polygon, when oriented, comprises a SAW from $\mathbf{0}$ to some neighbour v of $\mathbf{0}$, together with the edge $\langle v, \mathbf{0}\rangle$. A SAW (respectively, polygon) is said to have n steps if it has exactly n edges. Let c_n be the number of n-step SAWs from $\mathbf{0}$, and let p_n be the number of n-step polygons containing $\mathbf{0}$. We shall also be interested in 'bridges', which will be defined soon.

By the above definition of a polygon, we have that

$$2p_n = c_{n-1}(\partial \mathbf{0}) \leq c_n, \qquad n \geq 3, \tag{2.1}$$

where $c_m(\partial\mathbf{0})$ denotes the number of m-step SAWs from $\mathbf{0}$ ending at some neighbour of $\mathbf{0}$. We seek here conditions on G for which the exponential growth rates of c_n and p_n are equal, and towards this end we introduce the concept of a graph height function.

Definition 2.1. [10, Defn 3.1] *Let $G = (V, E) \in \mathcal{G}$ with root labelled $\mathbf{0}$. A* graph height function *(abbreviated to 'ghf') on G is a pair $(h, \mathcal{H})$ such that*

(a) *$h : V \to \mathbb{Z}$, and $h(\mathbf{0}) = 0$,*
(b) *$\mathcal{H} \leq \mathrm{Aut}(G)$ acts quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant*, in that*

$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$

(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

Note that, if $(h, \mathcal{H})$ is a ghf, then so is $(-h, \mathcal{H})$. Examples of graphs in $\mathcal{G}$ possessing a ghf are found in [10], and of graphs without a ghf in [9]. While possesssion of a ghf allows progress on SAWs and their so-called locality problem for connective constants, one needs more for the study of polygons.

Definition 2.2. *Let $(h, \mathcal{H})$ be a graph height function for $G \in \mathcal{G}$, and let $\rho \in \mathcal{H}$. We call $(h, \mathcal{H}, \rho)$ a* square graph height function *(abbreviated to 'square ghf') if*

(a) *ρ is a translation, in that it fixes no finite set F of vertices,*

(b) *ρ is height-preserving in that, for $v \in V$, we have $h(v) = h(\rho(v))$,*
(c) *ρ commutes with every $\alpha \in \mathcal{H}$.*

The term 'square ghf' should not be confused with the 'strong ghf' of [9], and it is motivated as follows. Under the conditions of the definition, for a non-height-preserving $\alpha \in \mathcal{H}$, the action on V of the pair (ρ, α) is as the square lattice $\mathbb{Z}^2$. One might relax somewhat these conditions, but for simplicity we retain the above.

These definitions are utilized as follows. Firstly, it was proved in [10] that, if G has a ghf, then one may define the notion of a 'bridge' on G, and moreover the bridge growth rate β equals the connective constant μ. This is elaborated later in this section. Secondly, our main theorem, Theorem 3.2, asserts that a graph with a square ghf has the property that $\limsup p_n^{1/n} = \beta$ (which in turn equals μ).

We discuss bridges next. Bridges were introduced (but not by that name) in [16] in the context of the hypercubic lattice $\mathbb{Z}^d$. Let $(h, \mathcal{H})$ be a ghf for $G \in \mathcal{G}$. An *h-bridge* $(v_0, v_1, \dots, v_n)$ is a SAW on G satisfying

$$h(v_0) < h(v_m) \le h(v_n), \qquad 0 < m \le n.$$

Let b_n be the number of n-step h-bridges π starting at $v_0 = \mathbf{0}$. Using quasi-transitivity and subadditivity, it may be shown that the limit

$$\beta(h) := \lim_{n\to\infty} b_n^{1/n} \tag{2.2}$$

exists, and $\beta(h)$ is called the *h-bridge constant*.

Note that the definition (2.2) of $\beta(h)$ depends on the choice of graph height function h. It was noted above that, if $(h, \mathcal{H})$ is a ghf, then so is $(-h, \mathcal{H})$. It turns out that $\beta(h) = \beta(-h)$ if $\mathcal{H}$ is what is called 'unimodular', and in this case the *bridge constant* β is defined to be their common value. In the non-unimodular case, we define β by $\beta = \max\{\beta(h), \beta(-h)\}$. In each case we have that $\beta = \mu$, whence $\beta = \beta(G)$ is independent of the choice of ghf. The above is summarised in the following theorem.

Theorem 2.3. *Let $G \in \mathcal{G}$ possess a ghf $(h, \mathcal{H})$, and denote by $\mu(G)$ the connective constant of G.*

(a) [10] *If $\mathcal{H}$ is unimodular, then $\beta(h) = \beta(-h)$, and their common value $\beta(G)$ satisfies $\beta(G) = \mu(G)$.*
(b) [21] *If $\mathcal{H}$ is non-unimodular, then $\beta(G) := \max\{\beta(h), \beta(-h)\}$ satisfies $\beta(G) = \mu(G)$.*

For discussions of unimodularity see [22, Sect. 8.2] or [10, Sect. 3].

The *growth function* of a transitive graph G is defined as

$$\Gamma_n = |\{v \in V : d(0, v) \le n\}|,$$

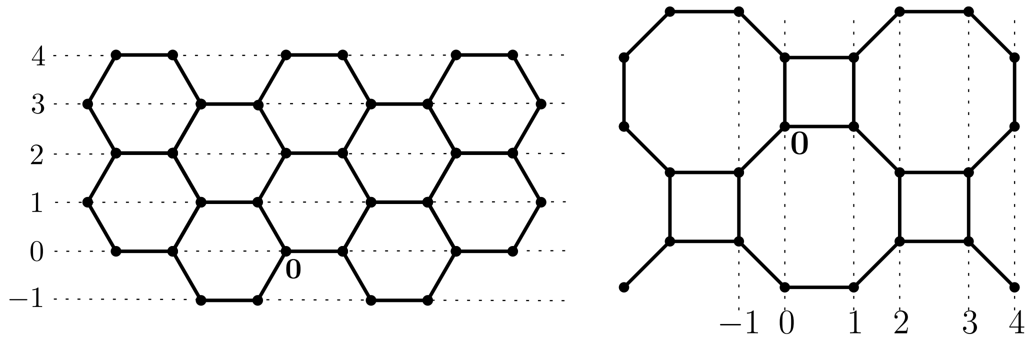

FIGURE 3.1. The hexagonal lattice and the square/octagon lattice. The heights of vertices are as marked. The automorphism ρ is a suitable shift rightwards for the first, and a suitable shift upwards for the second.

that is, the number of vertices in a ball of radius n. The graph is said to have *sub-exponential growth* if

$$\lim_{n\to\infty} \frac{1}{n} \log \Gamma_n = 0. \tag{2.3}$$

Remark 2.4. *It is useful to recall that any quasi-transitive graph G with sub-exponential growth is amenable, and hence any automorphism group of such G that acts quasi-transitively is unimodular (see* [27] *and* [22, Exer. 8.30]*).*

3. Main theorem

For a rooted graph G, let

$$\pi = \pi(G) := \limsup_{n\to\infty} p_n^{1/n}, \tag{3.1}$$

where p_n is the number of n-step polygons of G containing the root $\mathbf{0}$. The three main characters of this article are $\mu(G)$, $\beta(G)$, $\pi(G)$.

Remark 3.1. *Recall that, for quasi-transitive graphs $G \in \mathcal{G}$, $\mu(G)$ is independent of the choice of root (see* [13]*). Furthermore, for such G that in addition possess a ghf, $\beta(G)$ exists and is independent of the choice of root (see* [10, 21] *and Theorem 2.3). Subject to the conditions of the following theorem, $\pi(G)$ is also independent of the choice of root.*

Theorem 3.2. *Let $G \in \mathcal{G}$ have a square ghf $(h, \mathcal{H}, \rho)$ and sub-exponential growth.*

(a) *We have that $\pi(G) = \beta(G) = \mu(G)$.*
(b) *Furthermore, for $\epsilon > 0$, there exists an arithmetic sequence $(m_N : N \geq 1)$ of integers along which $\liminf_{N\to\infty} p_{m_N}^{1/m_N} \geq \beta(G) - \epsilon$.*

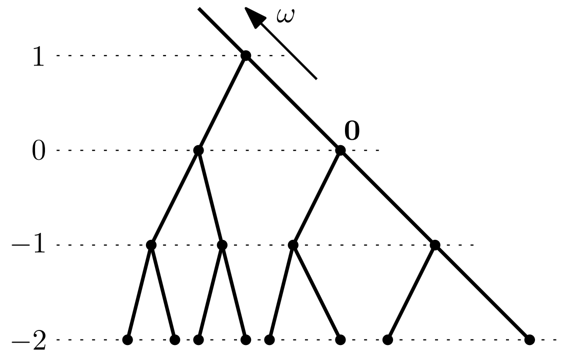

FIGURE 3.2. The binary tree with the 'horocyclic' height function.

The limsup of $p_n^{1/n}$ is identified in Theorem 3.2, but the story of its liminf is incomplete. The missing element seems to be a proof of the convergence of $p_n^{1/n}$. The proof of this for $\mathbb{Z}^d$ uses a fairly simple geometric construction together with subadditivity; this is due to Hammersley [14], but see also [23, Thm 3.2.3]. The geometry is however more challenging in the generality of the current paper. There exist nevertheless graphs satisfying the conditions of Theorem 3.2 to which the subadditivity argument may be adapted, so long as a certain extra condition on the ghf holds. We do not investigate this here.

It is a tautology that a bipartite graph G has no odd polygons. Thus, for bipartite graphs (such as the hypercubic, hexagonal, and square/octagon lattices), the liminf is taken along the even integers only.

By Theorem 3.2, the exponential growth rates of polygon and SAW counts are equal (under the stated conditions). This is much weaker than proving concrete polynomial bounds on p_n/c_n, as may be found in part for $\mathbb{Z}^d$ in [4, Thm 1.1].

The conditions of the theorem are illustrated by some examples.

Example 3.3. [10, Sect. 3] *Many (Euclidean) planar lattices satisfy the conditions of Theorem 3.2. We mention two examples, namely the hexagonal lattice and the square/octagon lattice as illustrated in Figure 3.1.*

Example 3.4. [10, Sect. 3] *The binary (that is, degree-3) tree $\mathbb{T}_3$ has exponential growth, and possesses a ghf but no square ghf, as follows. Let ω be a ray of $\mathbb{T}_3$, and 'suspend' $\mathbb{T}_3$ from ω (as illustrated in Figure 3.2). A given vertex on ω is labelled $\mathbf{0}$ and has height $h(\mathbf{0}) = 0$, and other vertices have their horocyclic heights, that is, their generation numbers relative to $\mathbf{0}$.*

Let $\mathcal{H}$ be the set of automorphisms of $\mathbb{T}_3$ that fix the end of $\mathbb{T}_3$ determined by ω. Then $(h, \mathcal{H})$ is a ghf. It follows (and is in fact trivial) that $\beta(\mathbb{T}_3) = \mu(\mathbb{T}_3) = 2$. On the other hand, $\mathbb{T}_3$ has no square ghf and, since it has no cycles, we have $\pi(\mathbb{T}_3) = 0$.

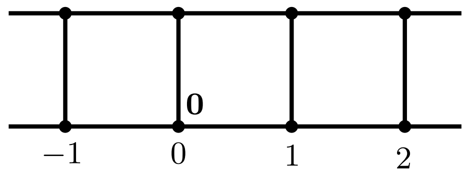

FIGURE 3.3. The ladder graph $\mathbb{L}$ with the root and heights indicated.

Example 3.5. [8, Sect. 3] *The ladder graph* $\mathbb{L} := \mathbb{Z} \times \{0, 1\}$ *of Figure 3.3 has subexponential growth, and possesses a ghf (as indicated in the figure) but no square ghf. Furthermore,* $\beta(\mathbb{L}) = \mu(\mathbb{L}) = \phi$ *and* $\pi(\mathbb{L}) = 1$*, where* $\phi = \frac{1}{2}(1 + \sqrt{5}) = 1.618\ldots$ *is the golden mean.*

Now consider the 'triple ladder' $\mathbb{L}_3 := \mathbb{Z} \times \{0, 1, 2\}$*. By Theorem 2.3,* $\beta(\mathbb{L}_3) = \mu(\mathbb{L}_3)$*. For simplicity, suppose a polygon* p *goes rightwards from* $\mathbf{0}$*. It forms a bridge of* $\mathbb{L}$ *(viewed as a subgraph of* $\mathbb{L}_3$*) until it reverses direction. Then it follows a leftwards bridge on a copy of* $\mathbb{L}$*, before once again turning back towards the origin (the shapes of the latter bridges are constrained by those of the earlier). The number of such* n*-step polygons is no larger than the number of* n*-step bridges of* $\mathbb{L}$ *(disregarding minor terms). Therefore,* $\pi \le \beta(\mathbb{L})$*. However,* $\beta(\mathbb{L}) < \beta(\mathbb{L}_3) = 1.914\ldots$*, (*[1, p. 198]*). This may be extended to wider ladders of the form* $\mathbb{L}_m := \mathbb{Z} \times \{0, 1, \ldots, m\}$*.*

Example 3.6. [12, 17] *Let* $G = \mathbb{T}_3 \times \mathbb{Z}$ *be the (Cartesian) product of a binary tree* $\mathbb{T}_3$ *and the doubly-infinite line* $\mathbb{Z}$*; vertices of* G *are expressed as vectors* (t, z)*, and two vertices* (t_1, z_1)*,* (t_2, z_2) *are adjacent if and only if either* $t_1 = t_2$ *and* $|z_1 - z_2| = 1$*, or* $t_1 \sim t_2$ *in* $\mathbb{T}_2$ *and* $z_1 = z_2$*. With* $(h, \mathcal{H})$ *as in Example 3.4, we define the ghf* $(h', \mathcal{H}')$ *on* G *as follows. For* $\alpha \in \mathcal{H}$*, define* α' *by* $\alpha'(t, z) := (\alpha(t), z)$*, and let* ρ *be the shift given by* $\rho(t, z) = (t, z+1)$*. The set of all such* α'*, together with* ρ*, generates a subgroup* $\mathcal{H}'$ *of* $\mathrm{Aut}(G)$ *that acts transitively. We let* $h'(t, z) = h(t)$*, thus obtaining the required ghf* $(h', \mathcal{H}')$ *on* G*. It may be checked that* $(h', \mathcal{H}', \rho)$ *is a square ghf.*

The graph G *has exponential growth and possesses a square ghf, whence* $\beta(G) = \mu(G)$ *by Theorem 2.3. We do not know whether or not* $\pi(G) < \beta(G)$*. By* [26, Thm 4] *(see also Theorem 3.7), such strict inequality would imply ballisticity* (3.2) *for the random SAW on* G*, thereby verifying part of a conjecture of Hutchcroft* [17, p. 2803]*. Such questions may be posed more generally for the graphs* $\mathbb{T}_k \times \mathbb{Z}^d$ *(in the natural notation). For example, using a remark from* [17, p. 2804] *together with the forthcoming Theorem 3.7, we have that* $\pi(\mathbb{T}_k \times \mathbb{Z}) < \mu(\mathbb{T}_k \times \mathbb{Z})$ *for* $k \ge 6$*.*

We make some comments on the proof of Theorem 3.2. Proofs for the special case $\mathbb{Z}^d$ may be found in Hammersley [14], Kesten [19], and Madras–Slade [23]. The relevant properties of $\mathbb{Z}^d$ include reflection-invariance and translation-invariance, and these are used in varying degrees in the three proofs. In addition, the proof of

[23] seems to use the Euclidean geometry of $\mathbb{R}^d$. Certain aspects of translation-invariance are preserved in the generality of the current article via the assumption of the existence of a ghf. Reflection-invariance is however more problematic. This is overcome here by an argument that reverts to Hammersley's original proof from 1961, but which avoids the gap in that proof at [14, p. 520] (see below).

Whereas this paper is directed at the *equality* of π and μ, earlier work of Madras–Wu [24] and Panagiotis [26] is devoted to *strict inequality* (that is, $\pi < \mu$) for regular tilings of the hyperbolic plane. Such tilings have exponential growth. Panagiotis [26] proved also that, for any transitive graph G (and indeed more generally, see [26, Rem. 4.1]), such strict inequality implies that random SAW on G is *ballistic* (sometimes expressed as having 'positive speed') in the sense that there exists $c > 0$ such that

$$\mathbb{P}_n\big(d(\mathbf{0}, L_n) \le cn\big) \le e^{-cn}, \qquad n \ge 1, \tag{3.2}$$

where $\mathbb{P}_n$ is the uniform probability measure on the set of n-step SAWs from the root $\mathbf{0}$ of G, and L_n is the final endvertex of the randomly selected SAW. Hutchcroft [17] has proved ballisticity for any graph with a transitive, non-unimodular (sub)group of automorphisms. The strict inequality $\pi < \mu$ is believed to hold for all non-amenable, transitive graphs.

For clarity, we present an explicit statement of the relationship between strict inequality and ballisticity (without claiming any originality).

Theorem 3.7. *Let $G \in \mathcal{G}$ and let*

$$\pi'(G) = \limsup_{n\to\infty} \Big(\sup_{v \in V} p_{n,v}\Big)^{1/n}, \tag{3.3}$$

where $p_{n,v}$ is the number of n-step polygons that include v. Then $\pi'(G) < \mu(G)$ if and only if there exist $c > 0$ and $N \ge 1$ such that

$$\mathbb{P}_{n,v}\big(d(v, L_n) \le cn\big) \le e^{-cn}, \qquad n \ge N, \quad v \in V, \tag{3.4}$$

where $\mathbb{P}_{n,v}$ is the uniform probability measure on the set of n-step SAWs starting at v, and L_n is the final endvertex of the randomly selected SAW.

Since G is assumed quasi-transitive, the inner supremum of (3.3) is over a finite set. If the conditions of Theorem 3.2 are satisfied, then $\pi'(G) = \pi(G)$.

Proof. That $\pi' < \mu$ implies (3.4) for suitable c, N is a consequence of [26, Thm 4, Rem. 4.1][1]. Conversely, by (2.1) and (3.4),

$$p_{n+1,v} = \tfrac{1}{2} c_{n,v} \mathbb{P}_{n,v}\big(d(v, L_n) = 1\big) \le \tfrac{1}{2} c_{n,v} e^{-cn}, \qquad n \ge N, \quad v \in V,$$

[1] Panagiotis's proof is similar to (but independent of) a proof of Hammersley in [14, p. 520]. Furthermore, it corrects an error in the latter proof, where it is stated (in the language of that paper) that $\Phi(n)$ is the union of the $\Phi_\psi(n)$.

where $c_{n,v}$ is the number of n-step SAWs from v. On recalling Remark 3.1, we deduce that $\pi'(G) \le \mu(G)e^{-c}$ as required. □

By Theorems 3.2 and 3.7, random SAWs in the graphs G of Theorem 3.2 are not ballistic. That is of course weaker than showing they are sub-ballistic in the sense of [5, 20].

4. Proof of Theorem 3.2

We begin with some further notation. Let $(h, \mathcal{H}, \rho)$ be a square ghf on $G = (V, E) \in \mathcal{G}$. A SAW from u to v is called *stiff* if all of its vertices x, other than its endvertices, satisfy $h(u) < h(x) < h(v)$. We shall define (as in [10]) a certain integer $r = r(h, \mathcal{H})$. If $\mathcal{H}$ acts transitively, we set $r = 0$. Assume $\mathcal{H}$ does not act transitively, and let $r = r(\mathcal{H})$ be the infimum of all r such that the following holds. Let $o_1, o_2, \dots, o_M$ be representatives of the (finitely many) orbits of $\mathcal{H}$. For $i \ne j$, there exists a vertex $v_j \in \mathcal{H}o_j$ together with a stiff SAW $\nu(o_i, v_j)$ from o_i to v_j with length r or less. We choose such a SAW, and denote it $\nu(o_i, v_j)$ as above. We set $\nu(o_i, o_i) = \{o_i\}$.

It is proved in [10, Prop. 3.2] that

$$0 \le r \le (M-1)(2d+1) + 2, \tag{4.1}$$

where $M = |V/\mathcal{H}|$ and

$$d = \max\big\{|h(x) - h(y) : x, y \in V,\ x \sim y\big\}. \tag{4.2}$$

The constant r will be used later in this section. By (4.2),

$$d(x, y) \ge \frac{1}{d}|h(x) - h(y)|, \qquad x, y \in V. \tag{4.3}$$

We state a lemma next.

Lemma 4.1. *Let $G \in \mathcal{G}$ have square ghf $(h, \mathcal{H}, \rho)$.*

(a) *We have that, as $k \to \infty$, $d(v, \rho^k(v)) \to \infty$ uniformly in $v \in V$.*

(b) *There exists an integer δ such that*

$$d(v, \rho^k(v)) \le k\delta, \qquad v \in V, \quad k \ge 1.$$

Proof. (a) Suppose there exists v such that $d(v, \rho^k(v)) \not\to \infty$. Since G is locally finite, there exists $w \in V$ and a subsequence (k_i) such that $\rho^{k_i}(v) = w$ for all i. Thus ρ fixes the set of vertices $\{\rho^k(v) : j_1 \le k < j_2\}$, in contradiction of the assumption that ρ is a translation.

Next we prove uniformity of divergence. Firstly, we claim that, for given k, $d(v, \rho^k(v))$ is constant on orbits of $\mathcal{H}$. To see this, let $u \in \mathcal{H}v$ and find $\alpha \in \mathcal{H}$ such

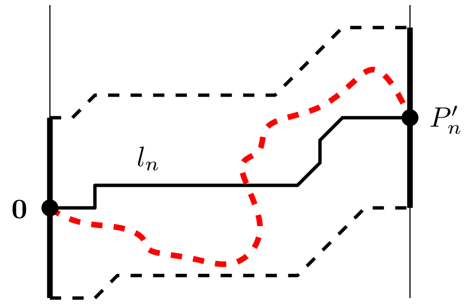

FIGURE 4.1. The region R_n. A shortest path l_n joins $\mathbf{0}$ to P_n', and the red (dashed) path is a bridge. The two vertical lines depict the sets of v such that $h(v) = 0$, and such that $h(v) = h(P_n')$, respectively.

that $v = \alpha(u)$. Since $\rho\alpha = \alpha\rho$,

$$d(v, \rho^k(v)) = d\big(\alpha(u), \rho^k(\alpha(u))\big) = d\big(\alpha(u), \alpha(\rho^k(u))\big) = d(u, \rho^k(u)). \tag{4.4}$$

Secondly, suppose $u \notin \mathcal{H}v$. By (4.1), there exists a path of length not exceeding r from u to some $w \in \mathcal{H}v$. By the triangle inequality,

$$\begin{aligned} d(u, \rho^k(u)) &\le d(u, w) + d(w, \rho^k(w)) + d(\rho^k(w), \rho^k(u)) \\ &\le 2r + d(w, \rho^k(w)) = 2r + d(v, \rho^k(v)) \qquad \text{by (4.4)}. \end{aligned} \tag{4.5}$$

The claimed uniformity follows from (4.4)–(4.5).

(b) By (4.4)–(4.5) with $k = 1$, $d(v, \rho(v))$ is bounded above by some $\delta > 0$, uniformly in v. By the triangle inequality,

$$d(v, \rho^k(v)) \le \sum_{l=0}^{k-1} d(\rho^l(v), \rho^{l+1}(v)),$$

and the inequality for general k follows. □

Let G satisfy the conditions of the theorem with square ghf $(h, \mathcal{H}, \rho)$. We write $\pi := \limsup_{n\to\infty} p_n^{1/n}$ as in (3.1). By (2.1) and Theorem 2.3,

$$\pi \le \mu = \beta. \tag{4.6}$$

Since G has sub-exponential growth, it is unimodular (see Remark 2.4). By Theorem 2.3(a), the bridge constant $\beta = \beta(G)$ satisfies $\beta = \beta(h)$.

Let $n \ge 1$ and recall the number b_n of h-bridges (henceforth called simply bridges). Find a vertex P_n with $h(P_n) \ge 1$ such that the number of bridges from $\mathbf{0}$ to P_n is at least $b_n/|\Gamma_n|$. Instead of working with the point P_n, we work with a point P_n' defined as follows. By the discussion leading to (4.1), there exists a vertex $P_n' \in \mathcal{H}\mathbf{0}$ such

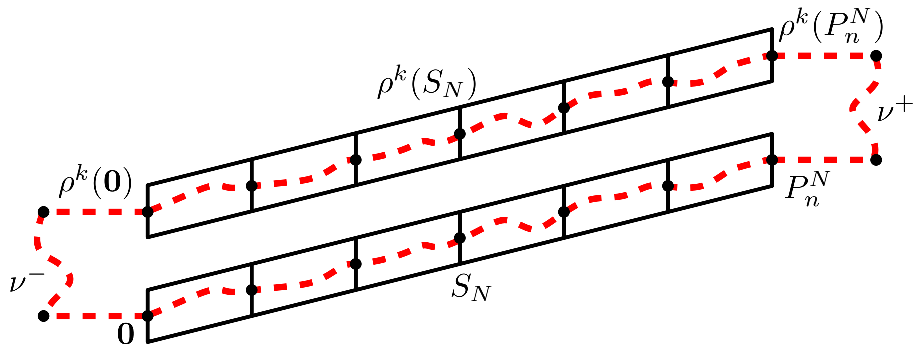

FIGURE 4.2. The tube S_N and its image $\rho^k(S_N)$. Each is traversed by an $(n+\ell)N$-step bridge, and these two bridges are joined into a red (dashed) polygon by adding the connecting paths $\nu^\pm$.

that there is a stiff SAW ν from P_n to P'_n with some length ℓ not exceeding r, We adjoin ν to any bridge from $\mathbf{0}$ to P_n to obtain that

(4.7) there exist at least $b_n/|\Gamma_n|$ bridges of length $n+\ell$ from $\mathbf{0}$ to P'_n.

We assume henceforth that $n \geq \ell$.

Let l_n be a shortest bridge from $\mathbf{0}$ to P'_n, and let R_n be the subgraph of G induced by the vertex-set

$$D_n := \{\mathbf{0}\} \cup \big\{v \in V : d(v, l_n) \leq n,\ 1 \leq h(v) \leq h(P'_n)\big\}.$$

Note that the bridges of (4.7) lie within R_n.

The region R_n is the basic ingredient of the following construction. Let $\gamma \in \mathcal{H}$ be such that $\gamma(\mathbf{0}) = P'_n$, so that γ maps R_n to an image $\gamma(R_n)$ with heights between $h(P'_n)$ and $2h(P'_n)$. We may think of the $\gamma^i(R_n)$, $i \in \mathbb{Z}$, as consecutive translates of R_n that fit together to form a 'tube' along which their bridges combine to create a longer bridge. See Figures 4.1 and 4.2.

Let $N \geq 1$ and let $S_N = \bigcup_{0 \leq i \leq N-1} \gamma^i(R_n)$ and $P_n^N = \gamma^{N-1}(P'_n)$. By (4.7),

(4.8) there exist at least $(b_n/|\Gamma_n|)^N$ $(n+\ell)N$-step bridges in S_N from $\mathbf{0}$ to P_n^N.

Let $l_{n,\infty}$ be the union $\bigcup_{-\infty<i<\infty} \gamma^i(l_n)$, considered as a doubly-infinite path. We now extend the S_N into a doubly-infinite tube containing $l_{n,\infty}$, by defining S_∞ to be the subgraph of G induced by the vertex-set

$$D_{n,\infty} = \big\{v \in V : d(v, l_{n,\infty}) \leq n\big\}.$$

The graph S_∞ is periodic in that $\gamma(S_\infty) = S_\infty$. Furthermore, S_N is a subgraph of S_∞.

Lemma 4.2. *There exists $k = k_n < \infty$ such that $S_\infty \cap \rho^k(S_\infty) = \varnothing$.*

Proof. Let $k \geq 1$, and suppose there exists $v \in S_\infty \cap \rho^k(S_\infty)$. Since $v \in \rho^k(S_\infty)$, we have $\rho^{-k}(v) \in S_\infty$, so that $v, \rho^{-k}(v) \in S_\infty$. Thus there exist two vertices of the form $u, \rho^k(u)$ lying in S_∞. We claim that, by Lemma 4.1(a), this cannot hold for sufficiently large k. To this end it suffices to show that

$$K := \sup\{d(v,w) : v, w \in S_\infty,\ h(v) = h(w)\}$$

satisfies

$$K < \infty. \tag{4.9}$$

We may express $l_{n,\infty}$ as the path $(\dots, z_{-1}, z_0 = \mathbf{0}, z_1, \dots)$ where $z_{-1} \in \gamma^{-1}(R_n)$ and $z_1 \in R_n$. Since $h(z_k) \to \pm\infty$ as $k \to \pm\infty$, we have by (4.3) that

$$\text{for } v \in V,\ d(v, z_k) \to \infty \text{ as } k \to \pm\infty. \tag{4.10}$$

By (4.10) and the periodicity of S_∞, there exists a finite subpath l' of $l_{n,\infty}$ such that

$$K \leq \sup\{d(v,w) : v, w \in V,\ d(v,l') \leq n,\ d(w,l') \leq n\}.$$

This is a supremum over a finite set, whence $K < \infty$, and the conclusion of the lemma follows by Lemma 4.1(a) for sufficiently large k. □

By Lemma 4.2, we may choose $k = k_n < \infty$ such that $S_\infty \cap \rho^k(S_\infty) = \varnothing$. Since $S_N \subseteq S_\infty$,

$$S_N \cap \rho^k(S_N) = \varnothing, \qquad N \geq 1. \tag{4.11}$$

By (4.8), there exist at least $(b_n/|\Gamma_n|)^N$ distinct bridges traversing each of S_N and $\rho^k(S_N)$, and we propose to join such bridges into polygons by adding connections between their endvertices, as illustrated in Figure 4.2.

Consider first a connection between $\mathbf{0}$ and $\rho^k(\mathbf{0})$ using vertices with negative height (apart from its endvertices). We construct three paths as follows.

(i) By Definition 2.1(c), we may find a path $(\mathbf{0}, c_1, c_2, \dots)$ such that $h(c_i) \leq -i$ for all i. By (4.3), we may choose $t = t_n$ such that $d(H_0, c_t) > k\delta$ where $H_0 = \{v \in V : h(v) = 0\}$ and δ is given in Lemma 4.1(b). We denote by ν_1 the path $(\mathbf{0}, c_1, c_2, \dots, c_t)$.

(ii) By Lemma 4.1(b), there exists a shortest path ν_2 from c_t to $\rho^k(c_t)$ with length not exceeding $k\delta$.

(iii) As above, the path $\nu_3 := \rho^k(\nu_1)$, reversed, joins $\rho^k(c_t)$ to $\rho^k(\mathbf{0})$.

The union of the ν_i contains a path ν^- from $\mathbf{0}$ to $\rho^k(\mathbf{0})$ whose vertices, apart from the two endvertices, have strictly negative heights. The length of ν^- does not exceed $2t_n + k\delta$.

By a similar construction we find a path ζ from P_n^1 to $\rho^k(P_n^1)$ using only vertices with heights strictly exceeding $h(P_n^1)$, except for its endvertices, and we set $\nu^+ = \gamma^{N-1}(\zeta)$. Thus ν^+ joins P_n^N to $\gamma^{N-1}(\rho(P_n^1)) = \rho(P_n^N)$, using only vertices (apart from the endvertices) with heights strictly exceeding $h(P_n^N)$. Let $l_n^\pm$ be the lengths of the above $\nu^\pm$, so that

$$l_n^\pm \le 2t_n + k\delta. \tag{4.12}$$

The above paths and bridges may be pieced together to form polygons. First we follow a bridge traversing S_N from $\mathbf{0}$ to P_n^N, followed by ν_n^+, followed by a bridge of $\rho^k(S_N)$ (reversed) from $\rho^k(\mathbf{0})$ to $\rho^k(P_n^N)$, and finally the path ν^- (reversed). It follows that

$$p_{2(n+\ell)N+l_n^-+l_n^+} \ge \left(\frac{b_n}{|\Gamma_n|}\right)^{2N}. \tag{4.13}$$

Take the $(2Nn)$th root to find that

$$p_{2(n+\ell)N+l_n^-+l_n^+}^{1/(2Nn)} \ge \left(\frac{b_n}{|\Gamma_n|}\right)^{1/n}. \tag{4.14}$$

Now, by (4.12),

$$\begin{aligned}\frac{2(n+\ell)N+l_n^-+l_n^+}{2nN} &\to 1+\frac{\ell}{n} \quad \text{as } N\to\infty\\ &\to 1 \qquad\quad \text{as } n\to\infty,\end{aligned} \tag{4.15}$$

whence $\pi \ge \beta$ by (2.2) and (2.3). This may be combined with (4.6) to obtain part (a) of the theorem.

We turn to the more specific part (b). Let $\epsilon > 0$ and choose $\epsilon', \eta > 0$ such that $(\beta-\epsilon')^{1-\eta} > \beta - \epsilon$. Pick n sufficiently large that

$$\frac{\ell}{n} < \eta, \qquad \left(\frac{b_n}{|\Gamma_n|}\right)^{1/n} > \beta - \epsilon',$$

and consider the arithmetic sequence $(m_N : N = 1, 2, \dots)$ where $m_N = 2(n+\ell)N + l_n^- + l_n^+$. By (4.14)–(4.15),

$$\liminf_{N\to\infty} p_{m_N}^{1/m_N} \ge (\beta-\epsilon')^{1-\eta} > \beta - \epsilon,$$

as required.

References

[1] S. E. Alm and S. Janson, *Random self-avoiding walks on one-dimensional lattices*, Comm. Statist. Stochastic Models **6** (1990), no. 2, 169–212. MR 1047969

[2] R. Bauerschmidt, H. Duminil-Copin, J. Goodman, and G. Slade, *Lectures on self-avoiding walks*, Probability and Statistical Physics in Two and More Dimensions, Clay Math. Proc., vol. 15, Amer. Math. Soc., Providence, RI, 2012, pp. 395–467. MR 3025395

[3] S. R. Broadbent and J. M. Hammersley, *Percolation processes. I. Crystals and mazes*, Proc. Cambridge Philos. Soc. **53** (1957), 629–641. MR 91567

[4] H. Duminil-Copin, A. Glazman, A. Hammond, and I. Manolescu, *On the probability that self-avoiding walk ends at a given point*, Ann. Probab. **44** (2016), no. 2, 955–983. MR 3474464

[5] H. Duminil-Copin and A. Hammond, *Self-avoiding walk is sub-ballistic*, Comm. Math. Phys. **324** (2013), no. 2, 401–423. MR 3117515

[6] P. Flory, *Principles of Polymer Chemistry*, Cornell University Press, 1953.

[7] G. R. Grimmett, *Probability on Graphs*, 2nd ed., Institute of Mathematical Statistics Textbooks, vol. 8, Cambridge University Press, Cambridge, 2018. MR 3751350

[8] G. R. Grimmett and Z. Li, *Bounds on connective constants of regular graphs*, Combinatorica **35** (2015), no. 3, 279–294. MR 3367126

[9] ______, *Self-avoiding walks and amenability*, Electron. J. Combin. **24** (2017), no. 4, Paper No. 4.38, 24 pp. MR 3743101

[10] ______, *Locality of connective constants*, Discrete Math. **341** (2018), 3483–3497. MR 3862645

[11] ______, *Self-avoiding walks and connective constants*, Sojourns in Probability Theory and Statistical Physics. III, Springer Proc. Math. Stat., vol. 300, Springer, Singapore, 2019, pp. 215–241. MR 4044395

[12] G. R. Grimmett and C. M. Newman, *Percolation in* $\infty + 1$ *dimensions*, Disorder in Physical Systems, Oxford Sci. Publ., Oxford Univ. Press, New York, 1990, pp. 167–190. MR 1064560

[13] J. M. Hammersley, *Percolation processes. II. The connective constant*, Proc. Cambridge Philos. Soc. **53** (1957), 642–645. MR 91568

[14] ______, *The number of polygons on a lattice*, Proc. Cambridge Philos. Soc. **57** (1961), 516–523. MR 123489

[15] J. M. Hammersley and K. W. Morton, *Poor man's Monte Carlo*, J. Roy. Statist. Soc. Ser. B **16** (1954), 23–38; discussion 61–75. MR 64475

[16] J. M. Hammersley and D. J. A. Welsh, *Further results on the rate of convergence to the connective constant of the hypercubical lattice*, Quart. J. Math. Oxford Ser. (2) **13** (1962), 108–110. MR 139535

[17] T. Hutchcroft, *Self-avoiding walk on nonunimodular transitive graphs*, Ann. Probab. **47** (2019), no. 5, 2801–2829. MR 4021237

[18] E. J. Janse van Rensburg, *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*, 2nd ed., Oxford Lecture Series in Mathematics and its Applications, vol. 39, Oxford University Press, Oxford, 2015. MR 3526233

[19] H. Kesten, *On the number of self-avoiding walks. II*, J. Mathematical Phys. **5** (1964), 1128–1137. MR 166845

[20] D. Krachun and C. Panagiotis, *Quantitative sub-ballisticity of self-avoiding walk on the hexagonal lattice*, (2023), https://arxiv.org/abs/2310.17299.

[21] C. Lindorfer, *A general bridge theorem for self-avoiding walks*, Discrete Math. **343** (2020), no. 12, 112092, 11 pp. MR 4137630

[22] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge University Press, Cambridge, 2016, http://pages.iu.edu/~rdlyons/. MR 3616205

[23] N. Madras and G. Slade, *The Self-Avoiding Walk*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2013. MR 2986656

[24] N. Madras and C. C. Wu, *Self-avoiding walks on hyperbolic graphs*, Combin. Probab. Comput. **14** (2005), no. 4, 523–548. MR 2160417

[25] W. J. C. Orr, *Statistical treatment of polymer solutions at infinite dilution*, Trans. Faraday Soc. **43** (1947), 12–27.

[26] C. Panagiotis, *Self-avoiding walks and polygons on hyperbolic graphs*, J. Graph Theory **106** (2024), no. 3, 435–473. MR 4750971

[27] P. M. Soardi and W. Woess, *Amenability, unimodularity, and the spectral radius of random walks on infinite graphs*, Math. Z. **205** (1990), no. 3, 471–486. MR 1082868

Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK

Email address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/