ON THE NUMBER OF CLUSTERS IN THE PERCOLATION MODEL

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ABSTRACT

The number of clusters per site in a percolation process on a finite region of a crystalline lattice converges strongly to a limit as the region expands to fill the space. This limit is a continuous function of the underlying probability. These results provide a partial justification of the method used by Sykes and Essam [7] in their calculation of exact values for critical probabilities of certain two-dimensional percolation processes.

1. *Introduction*

There is only one known method of calculating exact values for critical probabilities of percolation processes on non-trivial lattices. This is due to Sykes and Essam [7], who studied the expected numbers of clusters per site on a pair of "matching" lattices, and who showed that these two quantities exhibit a dual behaviour. Under certain reasonable but unverified assumptions they were able to deduce precise values for the critical probabilities of some two-dimensional processes. Their assumptions were twofold: that the expected number of clusters per site in a finite region converges to a limit as the region expands to fill the lattice, and that this limit, considered as a function of the underlying probability p of a bond or site being unblocked, has exactly one " singularity " which is at the critical value of p. They did not define this notion of singularity precisely—broadly speaking it is a phenomenon which occurs at the critical probability and which is observable by knowledge of the limit function. In this paper I justify the first of their assumptions, and describe some properties of the limit function; this may be seen as progress towards a complete justification of the method.

More specifically, I study the site percolation process on the square lattice, and prove that the number of clusters per site in a finite rectangle converges to a limit as the rectangle expands. This limit is a continuous function of the underlying probability p, and the convergence is with probability one and in any mean. This simple example is the most convenient process to study, but the same method will yield exactly analogous results for both bond and site processes on all crystalline lattices of any dimension. Of central importance in the proof is the observation that the number of clusters in a region is both " subadditive " and, when suitably modified, " superadditive ". These notions are akin to the one-dimensional processes of Hammersley and Welsh [**3**; pp. 61–110] and Kingman [**4**].

For a general survey of percolation theory see the review by Shante and Kirkpatrick [**5**]. The method of Sykes and Essam is described in more generality by Essam [**2**; pp. 197–270].

2. *The theorem*

We colour each site of the square lattice *black* with probability p $(0 \leqslant p \leqslant 1)$ independently of the colourings of all other sites; an uncoloured site remains *white*.

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This colouring process induces a subgraph of the lattice whose sites are those coloured black and whose bonds are those which join pairs of black sites. We shall consider the set of possible realizations of the process to be the set Ω of graphs obtainable in this manner. A *cluster* of a graph is a connected component.

Let $R((a, b), (m, n))$ denote the rectangular region of the square lattice containing the sites (x, y) satisfying $a \leqslant x < m, b \leqslant y < n$. The *restriction* of $\omega \in \Omega$ to R is the subgraph of ω obtained by deleting all sites not in R.

THEOREM. *Let $K_{mn}(\omega)$ denote the number of clusters of ω restricted to $R((-m, -n), (m, n))$. Then*

$$K_{mn}(\omega)/(4mn) \to \lambda \quad as \quad m, n \to \infty$$

in any mean and for almost all $\omega \in \Omega$, where

$$\lambda = \inf_{r,s} E(K_{rs})/(4rs) \tag{1}$$

is a continuous function of p.

The quantity $K_{mn}/(4mn)$ is the number of clusters per site of ω restricted to R. It is clear from (1) that the limit function $\lambda = \lambda(p)$ satisfies

$$\lambda(0) = \lambda(1) = 0, \quad \lambda(p) \geqslant 0.$$

Other information about λ may be deduced from relations (4) and (9), to be demonstrated in the proof of the theorem. In particular, these relations provide upper and lower bounds for $\lambda(p)$ which I will use to show that λ is continuous everywhere and differentiable at the origin. These bounds may also be used to calculate approximate values for $\lambda(p)$. For example, evaluation at $r = 1, s = 2$ yields

$$p-2p^2 \leqslant \lambda(p) \leqslant p-\tfrac{1}{2}p^2.$$

3. *Proof of the theorem*

For any $\omega \in \Omega$ and rectangle R let $C(R)$ denote the number of clusters of ω restricted to R. Then C is subadditive in the sense that for any collection $\mathcal{R} = \{R_i : i = 1, 2, \ldots, k\}$ of finite disjoint rectangles with union $R = \bigcup_i R_i$ we have that

$$C(R) \leqslant \sum_i C(R_i). \tag{2}$$

The process C satisfies a superadditive relation also:

$$C(R) \geqslant \sum_i C(R_i) - N, \tag{3}$$

where N is the number of bonds of ω which join pairs of sites belonging to different rectangles in $\mathcal{R}$. Inequality (3) holds because any bond which contributes to N belongs to exactly one of the $C(R)$ clusters in R, and the removal of this bond increases this quantity by at most one. We obtain (3) by progressive removal of all such bonds. Note also that C is an independent process in the sense that $\{C(R_i) : i = 1, 2, \ldots, k\}$ is a family of independent random variables.

Fix positive integers r and s. Any integers m and n may be expressed as

$$m = \alpha r + \beta \ (0 \leqslant \beta < r), \quad n = \gamma s + \delta \ (0 \leqslant \delta < s).$$

Let $C_{mn}(\omega)$ be the number of clusters of ω restricted to the rectangle $R\big((0, 0), (m, n)\big)$. This rectangle may be written as the disjoint union of the rectangles

$$\begin{aligned} R_{ij} &= R\Big(\big((i-1)r, (j-1)s\big), (ir, js)\Big), \quad 1 \leqslant i \leqslant \alpha, \quad 1 \leqslant j \leqslant \gamma \\ S_i &= R\Big(\big((i-1)r, \gamma s\big), (ir, \gamma s+\delta)\Big), \quad 1 \leqslant i \leqslant \alpha \\ T_j &= R\Big(\big(\alpha r, (j-1)s\big), \big(\alpha r+\beta, js\big)\Big), \quad 1 \leqslant j \leqslant \gamma \\ U &= R\big((\alpha r, \gamma s), (\alpha r+\beta, \gamma s+\delta)\big). \end{aligned}$$

Using (2) and the fact that $C(R)$ is no greater than the number of sites in R for any R, we deduce that

$$C_{mn} \leqslant \sum_{i=1}^{\alpha} \sum_{j=1}^{\gamma} C(R_{ij}) + \alpha\delta r + \gamma\beta s + \beta\delta.$$

But $\{C(R_{ij}) : i, j \geqslant 1\}$ is a collection of independent random variables distributed like C_{rs}; thus by the strong law of large numbers

$$\limsup_{m,\, n \to \infty} C_{mn}/(mn) \leqslant E(C_{rs})/(rs) \qquad \text{a.e.}$$

In particular, by Fatou's lemma,

$$\limsup_{m,\, n \to \infty} E(C_{mn})/(mn) \leqslant E(C_{rs})/(rs).$$

But this holds for any r and s, and hence

$$\lim_{m,\, n \to \infty} E(C_{mn})/(mn) = \inf_{r,\, s} E(C_{rs})/(rs).$$

Writing

$$\lambda = \lambda(p) = \inf_{r,\, s} E(C_{rs})/(rs), \tag{4}$$

we have proved that

$$P\Big(\limsup_{m,\, n \to \infty} C_{mn}/(mn) \leqslant \lambda\Big) = 1.$$

The proof of convergence will be essentially complete when we have shown that

$$P\Big(\liminf_{m,\, n \to \infty} C_{mn}/(mn) \geqslant \lambda\Big) = 1. \tag{5}$$

To see this note that (3) implies

$$L_{mn} \geqslant \sum_{i=1}^{\alpha} \sum_{j=1}^{\gamma} L(R_{ij}) + \sum_{i=1}^{\alpha} L(S_i) + \sum_{j=1}^{\gamma} L(T_j) + L(U), \tag{6}$$

where $L(R) = B(R) - C(R)$, $L_{mn} = L\Big(R\big((0, 0), (m, n)\big)\Big)$, and $B(R)$ is the number of bonds of ω which have exactly one endpoint in R and which exit from R in either a northerly or an easterly direction. We are unable to proceed by the ordinary strong law because the family $\mathscr{L} = \{L(R_{ij}) : i, j \geqslant 1\}$ contains dependent random variables. (This contrasts with the bond percolation process in which these variables are independent.) However $\mathscr{L}$ is an array of random variables whose joint distributions are invariant under the shifts

$$\sigma : L(R_{ij}) \to L(R_{i+1,\, j}), \quad \tau : L(R_{ij}) \to L(R_{i,\, j+1}),$$

and an ergodic theorem of Dunford [1] ensures that the limit

$$\xi = \lim_{\alpha,\,\gamma \to \infty} (\alpha\gamma)^{-1} \sum_{i=1}^{\alpha} \sum_{j=1}^{\gamma} L(R_{ij})$$

exists almost everywhere and satisfies

$$E(\xi) = E\big(L(R_{11})\big). \tag{7}$$

Actually ξ is constant (a.e.). For, let $s_1, s_2, \ldots$ be an ordering of the sites in the first quadrant of the square lattice and let $X_i(\omega)$ be the random colouring of s_i. It is not hard to see that ξ is $\mathscr{T}$-measurable, where $\mathscr{T}$ is the tail σ-field of the sequence $\{X_i : i \geqslant 1\}$ of independent random variables. But $\mathscr{T}$ contains only trivial events, and it follows from (7) that

$$P(\xi = E(L_{rs})) = 1.$$

Returning to (6) we observe that

$$L(S_i) \leqslant r+\delta, \quad L(T_j) \leqslant s+\beta, \quad L(U) \leqslant \beta+\delta;$$

hence

$$\limsup_{m,\,n \to \infty} L_{mn}/(mn) \leqslant E(L_{rs})/(rs) \qquad \text{a.e.} \tag{8}$$

By Fatou's lemma,

$$\lim_{m,\,n \to \infty} E(L_{mn})/(mn) = \inf_{r,\,s} E(L_{rs})/(rs)$$

which implies that

$$\begin{aligned} \lambda &= -\inf_{r,\,s} E(L_{rs})/(rs) \\ &= \sup_{r,\,s} \big(E(C_{rs}) - p^2(r+s)\big)/(rs). \end{aligned} \tag{9}$$

Also $L_{mn} \geqslant -C_{mn}$, and so by (8)

$$\begin{aligned} \liminf_{m,\,n \to \infty} C_{mn}/(mn) &\geqslant -E(L_{rs})/(rs) \qquad \text{a.e.} \\ &\to \lambda \quad \text{as} \quad r, s \to \infty. \end{aligned}$$

This establishes (5), and we deduce that

$$P\Big(\lim_{m,\,n \to \infty} C_{mn}/(mn) = \lambda\Big) = 1.$$

It follows immediately that

$$P\Big(\lim_{m,\,n \to \infty} K_{mn}/(4mn) = \lambda\Big) = 1 \tag{10}$$

by dissecting the rectangle $R\big((-m, -n), (m, n)\big)$ into four rectangles $R\big((-m, -n), (-1, -1)\big)$, $R\big((-m, 0), (-1, n)\big)$, $R\big((0, -n), (m, -1)\big)$, $R\big((0, 0), (m, n)\big)$, applying (2) and (3), and using the stationarity of the process $\{C_{mn}\}$. Convergence in any mean is a consequence of convergence almost everywhere because

$$0 \leqslant K_{mn}/(4mn) \leqslant 1.$$

Equations (4) and (9) provide bounds for $\lambda = \lambda(p)$:

$$0 \leqslant f_{rs}(p) - \lambda(p) \leqslant p^2(r^{-1} + s^{-1}), \text{ for any } r \text{ and } s, \tag{11}$$

where

$$f_{rs}(p) = E(C_{rs})/(rs),$$

which we now consider as a function of p. These bounds (11) are tight enough to demonstrate the continuity of $\lambda(p)$. For, given $\varepsilon > 0$, choose positive integers R and S such that $R^{-1} + S^{-1} < \frac{1}{2}\varepsilon$. Then by (11)

$$g_{RS}(p, p') - \tfrac{1}{2}\varepsilon \leqslant \lambda(p) - \lambda(p') \leqslant g_{RS}(p, p') + \tfrac{1}{2}\varepsilon \tag{12}$$

where

$$g_{rs}(p, p') = f_{rs}(p) - f_{rs}(p').$$

But f_{RS} is a finite polynomial in p. So there exists $h > 0$ such that $|g_{RS}(p, p')| < \frac{1}{2}\varepsilon$ whenever $|p - p'| < h$; the continuity of λ follows from (12). This completes the proof of the theorem.

Finally I show that λ is differentiable at the origin. Expanding f_{rs} as a polynomial in p we have that

$$f_{rs}(p) = p + o(p)$$

because the only term in p corresponds to the cases when only one site is black. This site may be any of the rs sites available, and it provides the only cluster. By (11)

$$p - p^2(r^{-1} + s^{-1}) + o(p) \leqslant \lambda(p) \leqslant p + o(p),$$

which yields

$$\lambda'(0) = 1.$$

Note. Professor Kingman has shown me a recent paper of Smythe [**6**], who proves limit theorems for multidimensional subadditive processes. These general results are not sufficient to treat the quantity which I have studied in this paper.

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