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Source: *Journal of Applied Probability*, Vol. 12, No. 4, (Dec., 1975), pp. 673-683
Published by: Applied Probability Trust
Stable URL: http://www.jstor.org/stable/3212718
Accessed: 15/08/2008 17:03

J. Appl. Prob. **12,** 673–683 (1975)
Printed in Israel

A NON-MARKOVIAN BIRTH PROCESS WITH LOGARITHMIC GROWTH

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Abstract

I show that the sum $S_n = \Sigma_{k=1}^n X_k$ of independent random variables converges in distribution when suitably normalised, so long as the X_k satisfy the following two conditions: $\mu(n) = E\,|X_n|$ is comparable with $E\,|S_n|$ for large n, and $X_k/\mu(k)$ converges in distribution. Also I consider the associated birth process $X(t) = \max\{n\colon S_n \leqq t\}$ when each X_k is positive, and I show that there exists a continuous increasing function $\nu(t)$ such that

$$\left.\begin{array}{l}\limsup\limits_{t\to\infty}\\ \\ \liminf\limits_{t\to\infty}\end{array}\right\} P(X(t)-\nu(t)<u) = \left\{\begin{array}{l}P(Y<u)\\ \\ P(Y+1<u)\end{array}\right.$$

for some variable Y with specified distribution, and for almost all u. The function ν satisfies $\nu(t) = A(1+o(t))\log t$. The Markovian birth process with parameters $\lambda_n = \lambda^{-n}$, where $0 < \lambda < 1$, is an example of such a process.

BIRTH PROCESS; CONVERGENCE IN DISTRIBUTION; SUMS OF INDEPENDENT RANDOM VARIABLES

1. Sums of independent random variables

Let $\{X_k\colon k = 1, 2, \cdots\}$ be a sequence of independent random variables with partial sums

$$S_n = \sum_{k=1}^{n} X_k.$$

Under certain general conditions the theory of the limiting distribution of the normalised sum

$$S'_n = \frac{S_n}{b_n} - a_n$$

is well established. It is customary to suppose that the random variables

$$X_{nk} = \frac{X_k}{b_n} - \frac{a_n}{n} \qquad (1 \leqq k \leqq n)$$

in the sum

Received 18 April 1975.

$$S'_n = \sum_{k=1}^{n} X_{nk}$$

are *uniformly asymptotically negligible* (u.a.n.), which is to say that for any $\varepsilon > 0$

$$\sup_k P(|X_{nk}| > \varepsilon) \to 0 \quad \text{as } n \to \infty,$$

and Gnedenko and Kolmogorov [2] study such sums in detail. Little is known in general about sums of random variables which are not u.a.n., and it is with such a sequence that I am concerned here.

Henceforth I shall suppose that the X_k satisfy

$$\mu(k) = E\,|X_k| < \infty, \quad \text{and} \tag{1}$$

$$\mu(k)/\mu(k+1) \to \lambda, \quad (0 < \lambda < 1) \quad \text{as} \quad k \to \infty.$$

$$X_k/\mu(k) \to L, \quad \text{as} \quad k \to \infty, \tag{2}$$

in distribution, for some random variable L.

The limiting distribution of

$$S'_n = S_n/\mu(n) \tag{3}$$

is given by the following theorem.

Theorem 1. *As* $n \to \infty$

$$S'_n \to Z \tag{4}$$

in distribution, where

$$Z = \sum_{k=0}^{\infty} \lambda^k L_k \tag{5}$$

and the L_k $(0 \leqq k < \infty)$ *are independent random variables distributed like* L.

What distributions may Z, defined by (5), have? Jessen and Wintner [5] have proved that Z is of pure type, which is to say that its distribution function is either absolutely continuous, singular or purely discontinuous. Clearly the distribution function of Z may be any of these three. If L is constant (a.e.) then so is Z. If L is Bernoulli taking values 0 and 1, each with probability $\frac{1}{2}$, then Z is uniformly distributed in $[0, 1]$ if $\lambda = \frac{1}{2}$, and $\frac{2}{3} Z$ has the Cantor distribution if $\lambda = \frac{1}{3}$. At any rate, a theorem of Lévy [6] ensures that Z has a continuous distribution function if and only if L is not constant (a.e.). Note that Z is normally distributed if and only if L is normally distributed. It is obvious that the latter condition is sufficient. Its necessity follows from a theorem of Cramér ([1], p. 525).

The normalising factor in Equation (3) is unusual in the sense that it is more common to divide by some function of the sum S_n itself. However $\mu(n)$ is comparable with the quantity

$$M(n) = \sum_{k=1}^{n} \mu(k) \tag{6}$$

in that

$$\mu(n)/M(n) \to 1 - \lambda \qquad \text{as} \quad n \to \infty.$$

This is shown by the next lemma, which I shall need in the proof of Theorem 1.

Lemma. For any real sequence $\{x_k\}$ *with partial sums* $s_k = \Sigma_{k=1}^{n} x_k$ *the following statements are equivalent*:

$$x_k / x_{k+1} \to \lambda \qquad (0 < \lambda < 1) \quad as \quad k \to \infty \tag{a}$$

$$x_n / s_n \to 1 - \lambda \qquad as \quad k \to \infty \tag{b}$$

$$s_n / s_{n+1} \to \lambda \qquad (0 < \lambda < 1) \quad as \quad k \to \infty. \tag{c}$$

It follows that the conclusion of Theorem 1 may be amended to read

$$S_n / M(n) \to (1 - \lambda) Z \qquad \text{as} \quad k \to \infty$$

in distribution.

It is easy to deduce the next result from Theorem 1.

Corollary 2. If $r > 1$ *is given, and* $\{X_k \colon k = 1, 2, \cdots\}$ *is a sequence of independent random variables which satisfy*

$$\alpha(k) = (E \mid X_k \mid^r)^{1/r} < \infty, \text{ and} \tag{7}$$

$$\alpha(k)/\alpha(k+1) \to \lambda \qquad (0 < \lambda < 1) \quad as \quad k \to \infty$$

$$X_k / \alpha(k) \to K, \text{ as } k \to \infty, \text{ in distribution,} \tag{8}$$

then

$$S_n / \alpha\,(n) \to \sum_{k=0}^{\infty} \lambda^k K_k \qquad as \quad k \to \infty \tag{9}$$

in distribution, where the K_k *are independent random variables distributed like* K.

2. A birth process

Suppose now that $\{X_k \colon k = 1, 2, \cdots\}$ is a sequence of positive random variables which satisfy Equations (1) and (2). The distribution of the associated birth process

$$X(t) = \max\{n\colon S_n \leqq t\}$$

may be studied through the distributions of the S_n, because $X(t) = n$ if and only if $S_n \leqq t < S_{n+1}$. I will use this simple relation to study the distribution of $X(t)$ for large t.

Theorem 3. *If L is not constant (a.e.), there exists a continuous increasing function $\nu(t)$ such that*

$$\left.\begin{matrix}\limsup_{t\to\infty} \\ \\ \liminf_{t\to\infty}\end{matrix}\right\} P(X(t) - \nu(t) < u) = \left\{\begin{matrix} P(Y < u) \\ \\ P(Y + 1 < u)\end{matrix}\right. \tag{10}$$

for all u, and for some random variable Y with specified distribution.

The variable Y in Theorem 3 is defined by

$$Y = -\log_\mu Z \tag{11}$$

where Z is given by (5), and $\mu\lambda = 1$. Y is finite almost everywhere because the distribution function of Z is continuous by the comment after the statement of Theorem 1. The function $\nu(t)$ is not necessarily unique, although the points t at which $\nu(t)$ is integer-valued are specified by the process X. The method of construction of ν in the proof ensures that

$$\nu(t) = (1 + o(t))\log_\mu t.$$

This theorem is best possible in the sense that $X(t) - \nu(t)$ converges almost nowhere because it has infinitely many jump discontinuities of unit size in any infinite time interval with probability one. Note however that X is an honest birth process if and only if L is not identically zero. This is an immediate consequence of Theorem 1 because

$$\begin{aligned} P(X(t) = \infty) &= \lim_{n\to\infty} P(S_n \leqq t) \\ &= P(Z \leqq 0)\end{aligned}$$

if the distribution function of Z is continuous at zero.

The case when $P(L = l) = 1$ $(l \geqq 0)$ raises few difficulties, since then Z is concentrated at the point $l/(1-\lambda)$, and a trivial modification of the proof of Theorem 3 shows that (10) holds so long as neither u nor $u - 1$ has the value $\log_\mu (1-\lambda)/l$. Hence

$$\lim_{t\to\infty} P(X(t) - \nu(t) < u) = \begin{cases} 0, & \text{if} \quad u < \log_\mu (1-\lambda)/l \\ \\ 1, & \text{if} \quad u > 1 + \log_\mu (1-\lambda)/l \end{cases}$$

and

$$\left.\begin{array}{l}\limsup_{t\to\infty} \\ \liminf_{t\to\infty}\end{array}\right\} P(X(t)-\nu(t)<u)=\left\{\begin{array}{l}1 \\ 0\end{array}\right. \quad \text{if} \quad 0<u-\log_{\mu}(1-\lambda)/l<1.$$

If L is concentrated at zero then $l=0$ and

$$\lim_{t\to\infty} P(X(t)-\nu(t)\geqq u)=1, \quad \text{for all } u.$$

Waugh [7] has studied birth processes of the same general type as I consider here, but he was interested in processes with increasing arrival rates.

Examples

(a) X_k is geometrically distributed as

$$P(X_k=j)=\lambda^j(1-\lambda^j)^k, \qquad 0\leqq j<\infty,$$

where $0<\lambda<1$. Then

$$\mu(k)=\mu^k-1,$$

where $\mu\lambda=1$, and

$$X_k/\mu(k)\to E \qquad \text{as} \quad k\to\infty$$

where E is negative exponential with parameter one. A possible choice for the function $\nu(t)$ in Theorem 3 is

$$\nu(t)=\log_{\mu}(t+1).$$

This is closely related to a process occurring in the study of the chromatic number of a random graph (see [3], [4]).

(b) X_k is negative exponentially distributed as

$$P(X_k>u)=\exp(-u\lambda_k), \qquad u\geqq 0,$$

where the λ_k satisfy

$$\lambda_k>0, \qquad \lambda_k/\lambda_{k+1}\to\lambda \qquad \text{as} \quad k\to\infty$$

and $0<\lambda<1$. Then

$$\mu(k)=\lambda_k^{-1}$$

and the distribution of $X_k/\mu(k)$ is negative exponential with parameter one for all k. A possible choice for $\nu(t)$ is the inverse of the function

$$\mu(x)=\lambda_n^{-1}(\lambda_{n+1}/\lambda_n)^{n-x}, \qquad x\geqq 1,$$

where $n=[x]$ is the integral part of x.

(c) X_k is distributed like $a_k X$ where X is some positive random variable with $E\,|\,X\,| < \infty$, and the a_k satisfy

$$a_k > 0, \qquad a_k/a_{k+1} \to \lambda, \qquad \text{as} \quad k \to \infty$$

and $0 < \lambda < 1$.

3. Proof of results

Proof of the Lemma.

$(a) \Rightarrow (b)$. Choose $\varepsilon > 0$ such that

$$0 < \lambda - \varepsilon < \lambda + \varepsilon < 1.$$

There exists N such that

$$\lambda - \varepsilon < \frac{x_k}{x_{k+1}} < \lambda + \varepsilon \qquad \text{whenever} \quad k \geqq N. \tag{12}$$

For $n \geqq m \geqq N$

$$s_n = s_{m-1} + s_{m,n} \tag{13}$$

where

$$s_{m,n} = \sum_{k=m}^{n} x_k$$

satisfies

$$\sum_{k=m}^{n} (\lambda - \varepsilon)^{n-k} \leqq \frac{s_{m,n}}{x_n} \leqq \sum_{k=m}^{n} (\lambda + \varepsilon)^{n-k} \tag{14}$$

by (12). Letting $n \to \infty$, $m \to \infty$ and $\varepsilon \downarrow 0$, in that order, (14) yields

$$\frac{s_{m,n}}{x_n} \to (1 - \lambda)^{-1}.$$

Equation (b) now follows immediately because

$$\frac{s_{m-1}}{x_n} \to 0 \quad \text{as} \quad n \to \infty$$

in (13).

$(b) \Leftrightarrow (c)$. This is immediate because

$$\frac{s_n}{s_{n+1}} = 1 - \frac{x_{n+1}}{s_{n+1}}. \tag{15}$$

$(b) \Rightarrow (a)$. We have that

$$\left(\frac{x_n}{s_n}\right) \Big/ \left(\frac{x_{n+1}}{s_{n+1}}\right) \to 1 \quad \text{as} \quad n \to \infty$$

and it follows from Equation (15) that

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{s_n}{s_{n+1}} = \lambda.$$

Proof of Theorem 1. First note that the sum (5) converges almost everywhere because the partial sums

$$Z_n = \sum_{k=0}^{n-1} \lambda^k L_k \tag{16}$$

form a semimartingale which is uniformly bounded in L^1 since

$$E\,|\,L\,| \leqq 1$$

by (2).

We shall need that the characteristic function

$$\phi(t) = E(\exp(itX))$$

of any random variable X satisfies

$$|1-\phi(t)| \leqq E\,|\,tX\,|. \tag{17}$$

To see this, let F be the distribution function of X, and write

$$\begin{aligned}|1-\phi(t)| &\leqq \int |1-e^{itu}|\,dF(u)\\ &= \int (2(1-\cos tu))^{\frac{1}{2}}\,dF(u).\end{aligned}$$

Equation (17) follows because

$$\cos tu \geqq 1 - \tfrac{1}{2}t^2u^2, \quad tu \geqq 0.$$

Let ϕ_k, ψ, f_n, g_m be the characteristic functions of X_k, L, S'_n, Z_m respectively. Then for $n \geqq m$

$$\begin{aligned}|f_n(t) - g_m(t)| \leqq |P_{m,n}(t)| \Big(&|1 - f_{n-m}(t_\mu (n-m)/\mu(n))| \\ &+ |1 - (g_m(t)/P_{m,n}(t))| \Big)\end{aligned} \tag{18}$$

where

$$P_{m,n}(t) = \prod_{k=n-m+1}^{n} \phi_k(t/\mu(n)). \tag{19}$$

It is clear that

$$\phi_{n-k}(t/\mu(n)) \to \psi(\lambda^k t) \quad \text{as } n \to \infty \tag{20}$$

by (1) and (2). Thus

$$P_{m,n}(t) \to g_m(t) \quad \text{as } n \to \infty,$$

and so, using (17),

$$\limsup_{n\to\infty} |f_n(t) - g_m(t)| \leqq |t| \limsup_{n\to\infty} (E|S_{n-m}|/\mu(n)). \tag{21}$$

But

$$E|S_{n-m}| \leqq \sum_{k=1}^{n-m} \mu(k)$$

which gives

$$\limsup_{n\to\infty} (E|S_{n-m}|/\mu(n)) \leqq \lambda^m (1-\lambda)^{-1} \quad \text{as } n \to \infty$$

by the Lemma. Hence

$$\lim_{m\to\infty} \limsup_{n\to\infty} |f_n(t) - g_m(t)| = 0. \tag{22}$$

Thus

$$\lim_{n\to\infty} f_n(t) = \lim_{m\to\infty} g_m(t) \tag{23}$$

as required to prove the theorem. It is easy to check that (23) follows on from (22). To see this, let

$$a_n = \mathrm{Re}(f_n(t)), \qquad b_m = \mathrm{Re}(g_m(t)).$$

be the real parts of f_n and g_m respectively. Then

$$\begin{aligned} 0 &= \lim_{m\to\infty} \limsup_{n\to\infty} |f_n(t) - g_m(t)| \\ &\geqq \lim_{m\to\infty} \limsup_{n\to\infty} |a_n - b_m| \\ &\geqq |\limsup_{n\to\infty} a_n - \lim_{m\to\infty} b_m| \\ &\geqq 0, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty} b_m.$$

But the same argument holds for $-a_n$ and $-b_m$, showing that $a_n - b_n \to 0$ as $n \to \infty$. Since this holds for the imaginary parts of f_n and g_m also, we have established the truth of (23), and thus completed the proof of the theorem.

Proof of Corollary 2. Equations (7) and (8) imply that

$$E\,|K|^r \leqq 1$$

and

$$E\,|X_k|/\alpha(k) \to E\,|K| \qquad \text{as } k \to \infty. \tag{24}$$

We may assume that $m = E\,|K|$ satisfies $m > 0$. A trivial modification is sufficient if this is not so. By (24), $\mu(k) = E\,|X_k|$ satisfies

$$\mu(k)/\alpha(k) \to m, \tag{25}$$

and so

$$\lim_{k\to\infty} \frac{\mu(k)}{\mu(k+1)} = \lim_{k\to\infty} \frac{\alpha(k)}{\alpha(k+1)}$$

$$= \lambda.$$

Also

$$X_k/\mu(k) \to K/m \qquad \text{as } k \to \infty$$

in distribution, by (8) and (25), and Theorem 1 tells us that

$$\frac{S_n}{\mu(n)} \to m^{-1} \sum_{k=0}^{\infty} \lambda^k K_k$$

in distribution. This gives the result by using (25) again.

Proof of Theorem 3. We may extend the domain of the function μ from the positive integers to the interval $[1, \infty)$ in such a way that the new function is continuous, strictly increasing, and satisfies

$$\lim_{x\to\infty} \frac{\mu(x)}{\mu(x+u)} = \lambda^u \tag{26}$$

for any u. An example of such an extension is given by

$$\mu(x) = \mu(n)\left(\frac{\mu(n+1)}{\mu(n)}\right)^{x-n}, \qquad x \geqq 1,$$

where $n = [x]$ is the integer part of x. The inverse function ν of μ is uniquely defined by

$$\nu(\mu(x)) = x, \qquad x \geqq 1;$$

ν is continuous and strictly increasing on the interval $[\mu(1), \infty)$.

For any u and all sufficiently large t

$$P(X(t) \geqq u + \nu(t)) = P(S_{[u+\nu(t)]} \leqq t). \tag{27}$$

Writing $s = \nu(t)$ we have that $t = \mu(s)$ and so

$$P(X(t) \geqq u + \nu(t)) = P\left(S'_{[u+s]} \leqq \frac{\mu(s)}{\mu([u+s])}\right), \tag{28}$$

where S'_n is defined by (3). But

$$\mu(u+s-1) \leqq \mu([u+s]) \leqq \mu(u+s), \tag{29}$$

and substitution into (28) yields

$$\begin{aligned} P(X(t) - \nu(t) \geqq u) &\geqq P\left(S'_{[u+s]} \leqq \frac{\mu(s)}{\mu(u+s)}\right) \\ &\to P(Z \leqq \lambda^{u}) \qquad \text{as } s \to \infty. \end{aligned} \tag{30}$$

The last step follows because the distribution function of Z is continuous, by the comment after the statement of Theorem 1, and by the fact that for any sequence $\{F_k : k = 1, 2, \cdots\}$ of distribution functions converging pointwise to the continuous distribution function F

$$F_k(x_k) \to F(x) \qquad \text{whenever } x_k \to x.$$

We deduce from (30) that

$$\limsup_{t\to\infty} P(X(t) - \nu(t) < u) \leqq P(Y < u) \tag{31}$$

where Y is defined by

$$Y = -\log_\mu Z.$$

Similarly, by (28) and (29),

$$\liminf_{t\to\infty} P(X(t) - \nu(t) < u) \geqq P(Y + 1 < u). \tag{32}$$

To show equality in relations (31) and (32) we evaluate the limit in (28) as s runs through the sequences $\{n - u : n = 1, 2, \cdots\}$ and $\{n + 1 - n^{-1} - u : n = 1, 2, \cdots\}$ respectively. This completes the proof.

It is clear that

$$\nu(t) = (1 + o(t)) \log_\mu t. \tag{33}$$

For

$$(\mu(x))^{1/x} \to \mu \qquad \text{as } x \to \infty$$

by (1) and (26). Substituting $x = \nu(s)$ and taking logarithms we obtain (33).

Finally note that

$$P(X(u\mu(r)) \geqq r) \rightarrow P(Z \leqq u)$$

as $r \rightarrow \infty$ through the integers.

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