

SAWS, TRAILS, KITES, PATTERNS

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ABSTRACT. A *self-avoiding walk* (SAW) on a transitive graph G is a path that visits no vertex more than once; a *trail* is a path that traverses no edge more than once. The number of n -step self-avoiding walks (respectively, trails) starting at a given vertex has order $\mu(G)^n$ (respectively, $\mu_T(G)^n$), where $\mu(G)$ is called the connective constant and $\mu_T(G)$ is the trail constant. It is immediate that $\mu = \mu_T$ if G either has degree 3 or less, or is a tree. We prove the strict inequality $\mu < \mu_T$ if G has degree 4 or more and is not a tree, and in addition G possesses a so-called transitive height function. The core of the proof is a pattern theorem for graphs possessing height functions, and this is achieved using an unpublished argument for $\mathbb{Z}^d$ by Hammersley, reported by Janse van Rensburg et al. (J. Stat. Phys. 85, 1996, 103–130). Some ancillary results are included concerning SAWs on the line graph of G .

1. INTRODUCTION

A *self-avoiding walk* (abbreviated to SAW) on a graph $G = (V, E)$ is a path that visits no vertex more than once. SAWs were introduced in the theory of polymerization (see Orr [38] and Flory [5]), and their critical behaviour has attracted the abundant attention since of mathematicians and physicists (see, for example, the book of Madras and Slade [33] and the lecture notes [3]).

Let $\mathcal{G}$ denote the set of all infinite, locally finite, (vertex-)transitive graphs without loops or parallel edges. For $G \in \mathcal{G}$ we designate some vertex the *root* of G , denoted $\mathbf{0}$. Let σ_n be the number of n -step SAWs on G starting at $\mathbf{0}$. It is a fundamental theorem of Hammersley [20] that the limit

$$\mu = \mu(G) := \lim_{n \rightarrow \infty} \sigma_n^{1/n}$$

exists, and the limit μ is called the *connective constant* of G . (The assumption of transitivity can be weakened to quasi-transitivity, [20].) It is proved in [13] that

$$\sqrt{d-1} \leq \mu(G) \leq d-1,$$

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where d is the vertex-degree of G .

SAWs are vertex-avoiding in that they visit no vertex twice or more. In contrast, a *trail* on $G \in \mathcal{G}$ is a path that is edge-avoiding in that it traverses no edge (irrespective of direction) twice or more. Let τ_n be the number of n -step trails starting at the root. The study of the sequence (τ_n) was initiated by Malakis [34, 35], who noted the existence of the ‘trail constant’

$$\mu_T = \mu_T(G) := \lim_{n \rightarrow \infty} \tau_n^{1/n}.$$

Since every SAW is a trail, we have $\mu \leq \mu_T$. This provokes the question of determining for which graphs the strict inequality $\mu < \mu_T$ holds. Guttmann [19] pointed out that $\mu = \mu_T$ if G has degree 3, since in that case (with the possible exceptions of the endpoints of the walk) a walk is vertex-avoiding if and only if it is edge-avoiding. It is immediate that $\mu = \mu_T$ if G is a tree, and thus we are led to the following question.

Question 1.1. *Let $G \in \mathcal{G}$. Is it the case that $\mu < \mu_T$ if and only if G has degree 4 or more and possesses a cycle?*

The main theorem of the current work is a partial resolution of this question.

Theorem 1.2. *Suppose $G \in \mathcal{G}$ has a transitive graph height function $(h, \mathcal{H})$. If G has degree 4 or more and possesses a cycle, then $\mu < \mu_T$.*

In principle, the proof allows a positive lower bound for $\mu_T - \mu$, but this is unlikely to be useful in practice.

A ‘graph height function’ (abbreviated henceforth as simply ‘height function’) of G is a function h that maps the vertex-set of G onto $\mathbb{Z}$ with certain properties that will be explained in Section 4. In brief, a height function is a formalization of the notion of the ‘heights’ of vertices, and it permits a SAW analysis based around so-called bridges. Height functions have been very useful in demonstrating properties of SAWs and connective constants on general vertex-transitive graphs. They were introduced in [15], and for further details the reader is referred to the review [16]. (For an elaboration of the assumption that the height function be transitive rather than merely quasi-transitive, see Section 6 of the current work.)

We outline next our approach to the proof of the Theorem 1.2. Suppose $G \in \mathcal{G}$ has degree 4 and possesses a cycle. A trail on G may enter a cycle, circumnavigate this cycle, and exit at the same point as it arrived. Such a local configuration constitutes a special type of ‘pattern’. The pattern theorem of Kesten [28], albeit restricted to SAWs on the hypercubic lattices $\mathbb{Z}^d$, suggests that such patterns should exist in all but a small proportion of n -step trails, and furthermore the latter proportion should be exponentially small in n . This is enough to imply $\mu(G) < \mu_T(G)$.

In this paper we present a pattern theorem for trails in a graph that possesses a height function. The proof does not follow the classic proof of Kesten [28], but

is instead motivated by an unpublished argument of Hammersley [22], as adapted in another setting by Janse van Rensburg et al. [27]. Hammersley's approach is in harmony with the existence of a height function since it is based on a renewal theory of bridges.

In Section 3, we present an account of what we believe may be not too distant from Hammersley's idea for patterns in $\mathbb{Z}^d$; this is included here both for historical reasons and also since it may be useful in other settings. Section 4 contains an extension of this new proof to a substantially more general family of graphs than the hypercubic lattices, namely transitive graphs possessing a height function. The SAW pattern theorem of that section has a cousin theorem for trails, proved similarly. It then remains to show in Section 5 that, for the graphs under study, all but an exponentially small proportion of trails contain a pattern that cannot belong to a SAW; the pattern in question is the circumnavigation of an entire cycle. This is followed in Section 6 by the 'pattern-theoretic' proof of Theorem 1.2.

The pattern theorem of Section 4 (Theorem 4.5) differs from that of Section 3 (Theorem 3.3) in that it applies to a much larger class of graphs than 'merely' the hypercubic lattices $\mathbb{Z}^d$. A further distinction is that, whereas Theorem 3.3 is proved using both *reflection* and *reversal* of walks, Theorem 4.5 is proved using *reversal* only.

There is some relationship between the results of this paper and those of [12]. The latter contains strict inequalities between SAW connective constants subject to perturbations in the underlying graph, by either adding new edges, or factoring by a normal subgroup of the automorphism group. Paper [12] proceeds by an algebraic adaptation of the pattern theorem, in the classical style of Kesten's proof from [28, 33]. In contrast, a principal purpose of the current paper is to highlight Hammersley's alternative approach (using bridges) to the pattern theorem.

Let $G \in \mathcal{G}$ have vertex-degree d . The line graph (or 'covering' graph) of G may be viewed as the graph obtained by replacing each vertex v of G by a copy of the complete graph K_d , each vertex of which is the mid-point of one of the edges incident to v . The line graph of G is denoted G_K in Section 7. When $d = 3$, G_K is the so-called 'Fisher graph' of G , as obtained by the 'Fisher transformation' of [4]. This transformation is a valuable tool in the study of a number of interactive models on graphs (see, for example, [11, 36]).

In an alternative approach to studying trails, one observes that a trail on $G \in \mathcal{G}$ gives rise to a certain SAW on its line graph G_K . However, when $d \geq 4$, not all SAWs on G_K correspond to trails on G . In Section 7, we include certain ancillary results concerning SAWs on line graphs, including a relation between the connective constants of G and G_K .

The methods of this paper may be applied to obtain strict inequalities for connective constants arising in certain other SAW-type problems. Here are two examples.

- (a) Let $k \geq 1$. Walks on G that visit no vertex more than k times.
- (b) Let $k, l \geq 1$. Walks $w = (w_0, w_1, \dots)$ on G such that $d_G(w_i, w_j) \geq k$ whenever $|i - j| \geq l$. Such walks may be required also to be self-avoiding. Here, d_G denotes graph-distance.

2. PRELIMINARIES

A graph $G = (V, E)$ has automorphism group $\text{Aut}(G)$, and it is called (*vertex*-) *transitive* if, for all $u, v \in V$, there exists $\alpha \in \text{Aut}(G)$ such that $\alpha(u) = v$. It is called *quasi-transitive* if $\text{Aut}(G)$ acts on V with only finitely many orbits, and *locally finite* if each vertex is incident to only finitely many edges. We denote by $\mathcal{G}$ the set of all infinite, locally finite, transitive graphs without loops or parallel edges.

For $G = (V, E) \in \mathcal{G}$, we call some vertex of V the *root* and denote it $\mathbf{0}$. An edge of E with endvertices u, v is written $\langle u, v \rangle$. If $\langle u, v \rangle \in E$, vertices u and v are called *neighbours*, and we write $u \sim v$. The *neighbourhood* ∂u of a vertex u is the set of its neighbours. Since our graphs are undirected, we identify $\langle u, v \rangle$ and $\langle v, u \rangle$. The *degree* $\deg(u)$ of vertex u is the number $|\partial u|$ of its neighbours. Since G is assumed transitive, $\deg(u)$ is constant for $u \in V$, and its common value is called the *degree* of G . The term $\mathcal{G}_d$ denotes the set of all $G \in \mathcal{G}$ with degree d .

A *walk* on G is an alternating sequence $w = (w_0, e_0, w_1, e_1, \dots)$ of vertices w_i and edges $e_i = \langle w_i, w_{i+1} \rangle$; since G has no parallel edges, we lose no information by suppressing mention of the edges e_i in the above listing of w . A walk with n edges is said to have n *steps*. The walk w is called *self-avoiding* if $w_i \neq w_j$ for all $i \neq j$, and is called a *trail* if $e_i \neq e_j$ for all $i \neq j$. A *cycle* is a walk $w = (w_0, w_1, \dots, w_n)$ such that $(w_0, w_1, \dots, w_{n-1})$ is self-avoiding and $w_0 = w_n$. The notation $\mathbb{Z}_k$ denotes a cycle of length k , that is, the Cayley graph of the cyclic group $\mathbb{Z}_k$.

Let σ_n be the number of n -step self-avoiding walks (or ‘SAWs’) from $\mathbf{0}$. It is standard that $\sigma_{m+n} \leq \sigma_m \sigma_n$, whence the limit $\mu(G) = \lim \sigma_n^{1/n}$ exists, and is called the *connective constant*.

Let $d \geq 2$. For $u \in \mathbb{Z}^d$ we write $u = (u_1, u_2, \dots, u_d)$. The *hypercubic lattice* $\mathbb{Z}^d$ has vertex-set $V = \mathbb{Z}^d$ and edge-set E containing all $\langle u, v \rangle$ satisfying $\sum_{i=1}^d |u_i - v_i| = 1$. The *root* (or origin) of $\mathbb{Z}^d$ is the zero-vector $\mathbf{0} = (0, 0, \dots, 0)$. The reader is referred to [33] for an account of SAWs on $\mathbb{Z}^d$, and to [16] for more general graphs.

When working on $\mathbb{Z}^d$, we define $h : \mathbb{Z}^d \rightarrow \mathbb{Z}$ by $h(x) = x_1$, the first coordinate of $x \in \mathbb{Z}^d$. Assume $n \geq 1$ in the following. A *half-space walk* $w = (w_0, w_1, \dots, w_n)$ on $\mathbb{Z}^d$ is a walk such $h(w_i) > h(w_0)$ for all $i \geq 1$. The SAW $w = (w_0, w_1, \dots, w_n)$ on $\mathbb{Z}^d$ is called a *bridge* if $w_0 = \mathbf{0}$ and $0 < h(w_i) \leq h(w_n)$ for $1 \leq i \leq n$. It is standard that the number b_n of n -step bridges on $\mathbb{Z}^d$ satisfies $b_{m+n} \geq b_m b_n$, and hence the limit $\beta(\mathbb{Z}^d) := \lim b_n^{1/n}$ exists, and is called the *bridge constant*. Furthermore,

$$(2.1) \quad b_n \leq \beta(\mathbb{Z}^d)^n, \quad n \geq 1.$$

We shall encounter some more general definitions in Section 4. For real numbers x, y we write

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}.$$

3. A PATTERN THEOREM FOR HYPERCUBIC LATTICES

Let $d \geq 2$, and consider the graph $\mathbb{Z}^d$ embedded in the usual way in $\mathbb{R}^d$. For $x \in \mathbb{R}$ we define the hyperplane $H_x = \{x\} \times \mathbb{R}^{d-1}$.

Definition 3.1. Let $e_k = \langle w_k, w_{k+1} \rangle$ be an edge of the walk $w = (w_0, w_1, \dots, w_n)$ and write $m_k = \frac{1}{2}(h(w_k) + h(w_{k+1}))$. The edge e_k is called a *cut-edge* of w if e_k is the unique edge of w that intersects H_{m_k} . A bridge is called *prime* if no edge except the first is a cut-edge.

Bridges of $\mathbb{Z}^d$ were defined in Section 2. Let $\mathcal{B}$ (respectively, $\mathcal{B}_p$) be the set of all bridges (respectively, prime bridges). A bridge may be decomposed into prime bridges, as explained next. Let $w = (w_0 = \mathbf{0}, w_1, \dots, w_n)$ be a bridge and let $e_k = \langle w_k, w_{k+1} \rangle$ be the earliest cut-edge of w other than e_0 (if no such cut-edge exists, we take $k = n$). Let $\alpha_1 \in \mathcal{B}_p$ be the sub-SAW from w_0 to w_k , noting that α_1 is a prime bridge. Let w' be the bridge obtained from the sub-SAW of w from w_k to w_n by translating it in such a way that its first vertex w_k is translated to the root; that is $w' = (w_k, w_{k+1} \dots, w_n) - w_k$. We may now express w in the form $w = \alpha_1 * w'$. By iteration, we find that $w = \alpha_1 * \alpha_2 * \dots * \alpha_r$ for prime bridges $\alpha_i \in \mathcal{B}_p$. Here, w is reconstructed from the α_i by a sequential augmentation: one starts with α_1 , and iteratively for each k , one augments the path π_{k-1} obtained from $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ by adding to it the SAW $\alpha_k + v_{k-1}$, where v_{k-1} is the endpoint of π_{k-1} other than the root $\mathbf{0}$.

We turn to so-called patterns. A SAW $p = (p_0, p_1, \dots, p_r)$, where $r \geq 1$, is called a *pattern* if it has no cut-edge (this definition may in fact be relaxed slightly but we retain it for later convenience). The pattern p is said to occur at the point w_i of the SAW $w = (w_0, w_1, \dots, w_n)$ if $i + r \leq n$ and $(w_i, w_{i+1}, \dots, w_{i+r}) = w_i + p - p_0$. The *number of occurrences* of p in w is the cardinality of the set of i such that p occurs at w_i . A pattern is called *bridgeable* if there exists some bridge w such that p occurs at least once in w . Let Σ_b denote the set of all bridgeable patterns.

The arguments of this section make use of the operations of reflection and reversal of SAWs. Let $\rho : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be the *reflection mapping* given by

$$\rho(x_1, x_2, \dots, x_d) = (-x_1, x_2, \dots, x_d).$$

For a SAW $w = (w_0, w_1, \dots, w_n)$ we define the reflected SAW $w_{\text{ref}} = \rho(w)$ and the reversed SAW $w_{\text{rev}} = R(w)$ by

$$(3.1) \quad w_{\text{ref}} := \rho(w) = (\rho(w_0), \rho(w_1), \dots, \rho(w_n)),$$

$$(3.2) \quad w_{\text{rev}} := R(w) = (w_n, w_{n-1}, \dots, w_0).$$

Note that ρ and R commute, and write

$$(3.3) \quad w_{\text{rr}} := R(\rho(w)).$$

It may be checked that, for a SAW w ,

$$(3.4) \quad \text{either all or none of } w, w_{\text{ref}}, w_{\text{rev}}, w_{\text{rr}} \text{ are patterns.}$$

We note an important fact.

Lemma 3.2. *Let b be a bridge. Any pattern p may occur in at most one of the prime bridges in the prime decomposition of b .*

Proof. This holds since such p has no cut-edge. $\square$

Theorem 3.3 (Pattern theorem). *Let $d \geq 2$ and let $p \in \Sigma_b$. Let $\sigma_n(P, \epsilon)$ be the number of n -step SAWs starting at $\mathbf{0}$ that contain at most ϵn copies of any of the elements of the set $P = \{p, p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}\}$. There exists $a > 0$ such that*

$$(3.5) \quad \limsup_{n \rightarrow \infty} \sigma_n(P, \epsilon)^{1/n} < \mu(G),$$

whenever $0 \leq \epsilon < a$.

The reason for including the patterns $p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}$ in the set P of (3.5) is as follows. As in [23], the proof has three stages (presented here in reverse order).

- (a) Using a geometrical construction, a SAW w is transformed into two half-space walks h_1, h_2 . In doing so, part of w is reversed.
- (b) Each half-space walk h_i is transformed into a bridge denoted b_i . This is done by decomposing h_i into bridges and ‘negative bridges’, and then reflecting the negative bridges to obtain ‘ordinary’ bridges.
- (c) The central part of the proof is to show that the number of n -step bridges containing no copy of p is exponentially small compared with the total number of n -step bridges.

In order that there be no copies of p in the bridges b_i , one needs to rule out copies of both p and p_{ref} in the half-space walks h_i . Working back through stage (a), one requires that w contain no copies of $p, p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}$.

Remark 3.4. *Whereas Theorem 3.3 makes use of both reflection and reversal, the more general Theorem 4.5 of Section 4 makes use of reversal but not reflection. When studying $\mathbb{Z}^d$ in this section, we choose to utilise the bridge arguments of Hammersley*

and Welsh [23] rather than the more general approach of Grimmett and Li [15] and Lindorfer [29]. In summary, the conclusion of the above theorem holds with the references to p_{ref} and p_{rr} removed throughout.

Remark 3.5 (Relationship to Kesten's pattern theorem). *Kesten's theorem, as presented in [33], applies to copies of a SAW p of $\mathbb{Z}^d$ for which there exists a SAW containing at least three copies of p (see [33, Prop. 7.1.3]). For graphs with a height function, a pattern satisfies the last condition if and only if it is bridgeable. Kesten's theorem makes no assumption on the cut-edges of P , whereas Theorem 3.3 requires there to be none, and additionally the latter bounds the number of appearances of members of the set $\{p, p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}\}$. As remarked in Remark 3.4, the reflected patterns p_{ref} , p_{rr} may be omitted here. The principal virtues of the current theorem compared to that of [28, 33] lie in the relative simplicity and clarity of its proof and its extension (given in the next section) to a class of graphs much larger than just $\mathbb{Z}^d$.*

Remark 3.6 (History). *The first pattern theorem was proved by Kesten in his influential work [28]. The proof given there is complicated, and may be found essentially in [33, Chap. 3]. That theorem has proved fundamental to the study of SAWs; see, for example, [1, 12, 27, 33]. Madras [32] has studied related but distinct combinatorial problems which require a different proof of the corresponding pattern theorem.*

The work [27] of Janse van Rensburg et al. is directed at self-avoiding ribbons rather than SAWs, and it uses a proof of the pattern theorem of which the main idea is attributed by the authors to Hammersley [22]. It was in reading [27] that the current author learned of Hammersley's argument.

Let $p \in \Sigma_b$, and let $b_n(p, 0)$ (respectively, $b_n(P, 0)$) be the number of n -step bridges that contain no copies of p (respectively, copies of none of the elements of the set $P = \{p, p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}\}$); we adopt a similar notation $(\sigma_n(p, 0)$, etc) in replacing bridges by SAWs. We have in the usual way (using Lemma 3.2) that

$$(3.6) \quad \sigma_{m+n}(P, 0) \leq \sigma_m(P, 0)\sigma_n(P, 0),$$

$$(3.7) \quad b_{m+n}(P, 0) \geq b_m(P, 0)b_n(P, 0),$$

so that, by the subadditive limit theorem, the following limits exists:

$$(3.8) \quad \mu(P) := \lim_{n \rightarrow \infty} \sigma_n(P, 0)^{1/n},$$

$$(3.9) \quad \beta(P) := \lim_{n \rightarrow \infty} b_n(P, 0)^{1/n},$$

$$(3.10) \quad \beta(p) := \lim_{n \rightarrow \infty} b_n(p, 0)^{1/n}.$$

By the definitions,

$$(3.11) \quad \mu(P) \leq \mu(p), \quad \beta(P) \leq \beta(p).$$

We state and prove an important proposition which is at the heart of Hammersley's approach.

Proposition 3.7. *Let $d \geq 2$ and let $p \in \Sigma_b$. We have the strict inequality*

$$(3.12) \quad \beta(p) < \beta(\mathbb{Z}^d),$$

and, moreover,

$$(3.13) \quad \beta(P) = \mu(P), \quad \beta(\mathbb{Z}^d) = \mu(\mathbb{Z}^d).$$

Proof. As noted above, a bridge w may be expressed uniquely as a concatenation of its successive prime elements, namely $w = \alpha_1 * \alpha_2 * \cdots * \alpha_r$. Let b_n be the number of n -step bridges and q_n the number of n -step, prime bridges. As in classical renewal theory (see, for example, [18, Chap. 10], [27]), we have that

$$(3.14) \quad b_n = q_n + \sum_{m=1}^{n-1} q_m b_{n-m}, \quad n \geq 1.$$

Therefore, the generating functions

$$B(x) = \sum_{\sigma \in \mathcal{B}} x^{|\sigma|}, \quad Q(x) = \sum_{\sigma \in \mathcal{B}_p} x^{|\sigma|},$$

satisfy

$$B(x) = Q(x) + Q(x)B(x),$$

whence

$$(3.15) \quad B(x) = \frac{Q(x)}{1 - Q(x)} \quad \text{whenever } Q(x) < 1.$$

By the definition of the bridge constant $\beta = \beta(\mathbb{Z}^d)$, B has radius of convergence $1/\beta$, so that, by Abel's lemma (see, for example, [18, p. 171]),

$$(3.16) \quad Q(\beta^{-1}) = 1.$$

Let $p \in \Sigma_b$, and let Q_p be the generating function of prime bridges containing no copy of p . Since p is assumed bridgeable, by (3.16), $Q_p(\beta^{-1}) < 1$. Since Q_p is continuous inside its ball of convergence, there exists $\eta = \eta(p) > 0$ such that

$$(3.17) \quad Q_p(\beta^{-1} + \eta) < 1.$$

We repeat the above for bridges containing no copies of p . By the argument leading to (3.15) adapted to such bridges, we deduce by (3.17) that $\beta(p)^{-1} > \beta^{-1} + \eta$, as required for (3.12). That is, the proportion of n -step bridges that contain no copies of p is exponentially small in n . That completes the novel element of this proof. The remaining part is fairly standard and the details are omitted in favour of a summary.

The second equality of (3.13) is well known (and was first proved by Hammersley and Welsh [23]). The reader is referred to the overview of their proof presented after the statement of Theorem 3.3. The proof details are omitted here, but may be found at [33, Cor. 3.1.8], [3, Sec. 2], [26, p. 436], and elsewhere; moreover, this argument is cited in [27, Lemma 3.1]. Using Lemma 3.2 and the invariance of P under reflection and reversal, it may be checked that the same argument may be applied to SAWs that contain no copies of any element of the set $P = \{p, p_{\text{ref}}, p_{\text{rev}}, p_{\text{rr}}\}$.

As in [33, Thm 3.1.1, eqn 3.1.8], we conclude that there exists $c = c(p, d) < \infty$ such that

$$(3.18) \quad \sigma_n(P, 0) \leq e^{c\sqrt{n}} \beta(P)^n$$

for all large n .

It is immediate that $\beta(P) \leq \mu(P)$, and the first equality of (3.13) follows by (3.18). $\square$

Proof of Theorem 3.3. By Proposition 3.7 and (3.11), we have

$$\mu(P) = \beta(P) \leq \beta(p) < \mu(\mathbb{Z}^d),$$

whence

$$\lim_{n \rightarrow \infty} \sigma_n(P, 0)^{1/n} < \mu(\mathbb{Z}^d).$$

The extended claim (3.5) follows in the same manner as [33, Lemma 7.2.5], to which readers desirous of a full proof are referred. $\square$

4. A PATTERN THEOREM FOR GENERAL CRYSTALS

The word ‘crystal’ in the section title is taken from [10, 21].

4.1. Height functions and bridges. Height functions were introduced under the title ‘graph height function’ in [15]. Since the distinction between ‘graph height function’ and ‘group height function’ is irrelevant here, we adopt the briefer term ‘height function’. First we define a height function, and then we use such functions to define bridges in general graphs.

Definition 4.1. [15, Defn 3.1] *Let $G = (V, E) \in \mathcal{G}$ have root labelled $\mathbf{0}$.*

- (i) *A height function on G is a pair $(h, \mathcal{H})$ such that:*
 - (a) *$h : V \rightarrow \mathbb{Z}$, and $h(\mathbf{0}) = 0$,*
 - (b) *$\mathcal{H} \leq \text{Aut}(G)$ acts quasi-transitively on G such that h is $\mathcal{H}$ -difference-invariant, in the sense that*

$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \quad \alpha \in \mathcal{H}, \quad u, v \in V,$$

- (c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

- (ii) A height function $(h, \mathcal{H})$ is called *transitive* (respectively, *unimodular*) if the action of $\mathcal{H}$ is transitive (respectively, unimodular).

Note that $(h, \mathcal{H})$ is a height function if and only if $(-h, \mathcal{H})$ is a height function. We shall sometimes omit explicit reference to the group $\mathcal{H}$.

Height functions have proved useful for several problems associated with SAWs, since they allow arguments based around bridges. For example, it was shown in [15, Thm 5.1] that a graph possessing a unimodular height function has connective constant with the property of locality (the condition of unimodularity was removed by the main theorem of [29]).

The existence, non-existence, and consequences of height functions were explored for various families of graphs in [10, 14, 15, 17]; see [16] for a review. See [30, Chap. 8] and [15, eqn (3.1)] for discussions of unimodularity. For clarity, we list some examples of graphs with or without a height function.

- (a) The hypercubic graphs $\mathbb{Z}^d$ have a transitive height function ([15]).
- (b) The hexagonal lattice has a quasi-transitive height function but no transitive height function ([15]).
- (c) The Cayley graphs of the first Grigorchuk group of [8] and the Higman group of [25] have no height function ([14, Thms 5.1, 8.1]).

Henceforth in this section, let $G \in \mathcal{G}$ and let $(h, \mathcal{H})$ be a height function for G . The *span* of a SAW $w = (w_0, w_1, \dots, w_n)$ on G is given by

$$(4.1) \quad \text{span}(w) = \max_i h(w_i) - \min_i h(w_i).$$

We turn to bridges. There are two types of bridge depending on whether we work with the height function h or the height function $-h$. A SAW $w = (w_0 = \mathbf{0}, w_1, \dots, w_n)$ with $n \geq 1$ is called a *(+)bridge* if

$$(4.2) \quad h(w_0) < h(w_i) \leq h(w_n), \quad 0 < i \leq n,$$

and a *(-)bridge* if

$$(4.3) \quad h(w_0) > h(w_i) \geq h(w_n), \quad 0 < i \leq n.$$

Definition 4.2. Let $e_k = \langle w_k, w_{k+1} \rangle$ be an edge of the walk $w = (w_0, w_1, \dots, w_n)$. The edge e_k is called a

$$\begin{aligned} (+)\text{cut-edge of } w \text{ if } h(w_i) & \begin{cases} \leq h(w_k) & \text{for } i \leq k, \\ > h(w_k) & \text{for } i > k, \end{cases} \\ (-)\text{cut-edge of } w \text{ if } h(w_i) & \begin{cases} \geq h(w_k) & \text{for } i \leq k, \\ < h(w_k) & \text{for } i > k, \end{cases} \end{aligned}$$

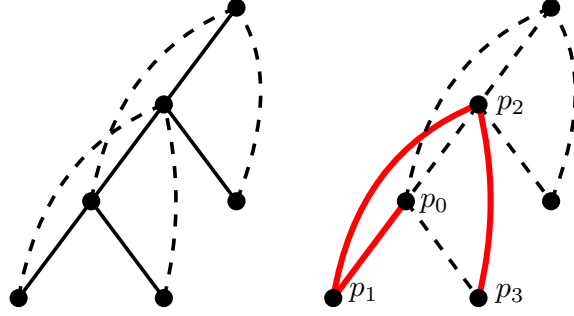


FIGURE 4.1. The grandparent graph of Trofimov [39] is derived from the binary tree by adding an edge from each vertex to its grandparent. Part of this graph is illustrated, with the pattern p indicated on the right.

A $(\pm)$ bridge $w = (w_0, w_1, \dots, w_n)$ is called *prime* if no edge except its first is a $(\pm)$ cut-edge.

Let $b_n^\pm$ be the number of n -step $(\pm)$ bridges. We write $\mathcal{B}^\pm$ (respectively, $\mathcal{B}_p^\pm$) for the set of $(\pm)$ bridges (respectively, prime $(\pm)$ bridges).

It may be shown ‘in the usual way’ (see [15, Sect. 4]) that the limits $\beta^\pm = \lim_{n \rightarrow \infty} (b_n^\pm)^{1/n}$ exist, and $\beta^\pm = \beta^\pm(G)$ is called the $(\pm)$ bridge constant of G . Moreover, it is proved in [15, Prop. 4.2] that there exists $r = r(G, h, \mathcal{H}) < \infty$ such that

$$(4.4) \quad b_n^\pm \leq \beta^\pm(G)^{n+r}, \quad n \geq 1.$$

(See also Lemma 5.4(a).) If the height function $(h, \mathcal{H})$ is transitive, we may take $r = 0$; we shall explain the value of r for the non-transitive case in Section 5.2. It is proved in [15, Thm 4.3] that $\beta^+ = \beta^-$ if the underlying graph G is unimodular. In addition the common value is independent of the choice of height function $(h, \mathcal{H})$, and moreover it equals $\mu(G)$; this is generally false when G is non-unimodular (see [29]).

The following was proved by Lindorfer [29] using an extension of the methods of [15, 23]. The set $\mathcal{Q}$ denotes the set of members of $\mathcal{G}$ enriched by relaxing the assumption of transitivity to quasi-transitivity.

Theorem 4.3 ([29, Thm 3.2]). *Let $G \in \mathcal{Q}$ possess a height function $(h, \mathcal{H})$. Then $\mu(G) = \beta^+(G) \vee \beta^-(G)$.*

Outlines of the proof of this theorem when the height function is transitive (respectively, quasi-transitive) are provided within the proofs of Section 5.

A *pattern* is defined as a SAW of length one or more that has neither a $(+)$ cut-edge nor a $(-)$ cut-edge. A pattern p is $(\pm)$ bridgeable if there exists a $(\pm)$ bridge w and

$\alpha \in \mathcal{H}$ such that w contains $\alpha(p)$. Let $\Sigma_b^\pm$ denote the set of $(\pm)$ bridgeable patterns, and $\Sigma_b = \Sigma_b^+ \cup \Sigma_b^-$. One may wonder whether $\mathcal{H}$ could be replaced by $\text{Aut}(G)$ in the previous sentence, but we note that bridges without images of p under $\text{Aut}(G)$ are rarer than those without images under $\mathcal{H}$.

Example 4.4. *Let p be a pattern. When $G = \mathbb{Z}^d$, it is not hard to see that $p \in \Sigma_b^+$ if and only if $p \in \Sigma_b^-$. This does not hold for all G , as the following indicates. Consider the grandparent graph G of Figure 4.1. Endow G with the horocyclic height function of [15, Rem. 3.3], and let p be the highlighted SAW. Then $p \in \Sigma_b^-$ but $p \notin \Sigma_b^+$.*

4.2. Pattern theorem for SAWs. Unlike the hypercubic lattices, general graphs do not usually have a property of reflection-invariance. The action of reflection is then replaced (as in [15]) by that of reversal. As in (3.2), the *reversal* of a SAW $w = (w_0, w_1, \dots, w_n)$ is the SAW

$$w_{\text{rev}} := (w_n, w_{n-1}, \dots, w_0),$$

and we write $w_{\text{rev}} = R(w)$.

We shall employ the notation of (3.8)–(3.10), and we recall (3.11). Here is our pattern theorem for graphs with height functions. It concerns general bridgeable patterns except those lying in a certain special set Π of SAWs satisfying

$$(4.5) \quad \Pi = \emptyset \text{ when the height function is transitive.}$$

When the height function is not transitive but quasi-transitive, the excluded set Π is a certain set of SAWs defined just prior to the forthcoming Example 5.3 and the relevant proof in Section 5.2. For definiteness, we note that

$$(4.6) \quad 0 \leq |\Pi| \leq M(M-1),$$

where M is the number of orbits of V under $\mathcal{H}$.

Theorem 4.5 (Pattern theorem for SAWs). *Let $(h, \mathcal{H})$ be a height function of $G \in \mathcal{G}$, and let $p \in \Sigma_b \setminus \Pi$.*

- (a) *We have that $\mu(p \cup p_{\text{rev}}, 0) < \mu(G)$.*
- (b) *Let $\sigma_n(p \cup p_{\text{rev}}, \epsilon)$ be the number of n -step SAWs starting at $\mathbf{0}$ that contain no more than ϵn copies of either p or p_{rev} . There exists $a > 0$ such that*

$$(4.7) \quad \limsup_{n \rightarrow \infty} \sigma_n(p \cup p_{\text{rev}}, \epsilon)^{1/n} < \mu(G),$$

whenever $0 \leq \epsilon < a$.

See Section 5 for the proof.

4.3. Pattern theorem for trails. Key to the proof of Theorem 1.2 is the observation that the pattern theorem for SAWs applies, *mutatis mutandis*, to trails. When replacing SAWs by trails, the notions of height function, bridge, half-space walk/trail, reversal/reflection, cut-edge, primality, bridgeability, and so on, carry over with no significant change. In order to distinguish between notation for SAWs and for trails, in the latter case we adorn relevant notation with the suffix ‘T’.

Let $G \in \mathcal{G}$ have height function $(h, \mathcal{H})$. A $(\pm)t$ -bridge (or ‘trail-bridge’) is a trail $w = (w_0, w_1, \dots, w_n)$ satisfying (4.2) or (4.3), as appropriate. A $(\pm)t$ -bridge is *prime* if it has no $(\pm)$ cut-edge apart from its first edge. A t -pattern (or ‘trail-pattern’) is a trail with length 1 or more containing neither $(+)$ cut-edge nor $(-)$ cut-edge. A t -pattern p is $(\pm)t$ -bridgeable if there exists a $(\pm)t$ -bridge w containing some image of p under $\mathcal{H}$. The set of $(\pm)t$ -bridgeable t -patterns is denoted $\Sigma_{T,b}^\pm$, and we let $\Sigma_{T,b} = \Sigma_{T,b}^+ \cup \Sigma_{T,b}^-$.

The proof of the following follows that of Theorem 4.5 and is omitted. The set Π , viewed as a set of trails, is as in Theorem 4.5, and satisfies (4.5) as there.

Theorem 4.6 (Pattern theorem for trails). *Let $(h, \mathcal{H})$ be a height function of $G \in \mathcal{G}$, and let $p \in \Sigma_{T,b} \setminus \Pi$.*

- (a) *We have that $\mu_T(p \cup p_{\text{rev}}, 0) < \mu_T(G)$.*
- (b) *Let $\tau_n(p \cup p_{\text{rev}}, \epsilon)$ be the number of n -step trails starting at $\mathbf{0}$ that contain no more than ϵn copies of either p or p_{rev} . There exists $a > 0$ such that*

$$(4.8) \quad \limsup_{n \rightarrow \infty} \tau_n(p \cup p_{\text{rev}}, \epsilon)^{1/n} < \mu_T(G),$$

whenever $0 \leq \epsilon < a$.

The proof of this theorem follows, *mutatis mutandis*, that of Theorem 4.5, and is omitted.

Example 4.7. *One naturally asks whether such a theorem holds for graphs without a height function. The obvious proof route would go via Kesten’s approach, a simple interpretation of which might be phrased as follows. “For any given t -pattern p and any sufficiently long trail t , one may perturb t locally to create a copy of p .” This assertion is problematic even for a graph with a quasi-transitive height function.*

*Consider the graph $G = \mathbb{Z}_m * K_2 * K_2$ of Figure 4.2 with, for concreteness, $m = 4$; here $\mathbb{Z}_m$ denotes an m -cycle and K_2 an edge. (See [6] for basic properties of free product graphs.) Let $\square$ denote a given 4-cycle of G and let v be a vertex of $\square$. The t -pattern $p(\square, v)$ is defined as the trail that circumnavigates $\square$ from v to v (the direction of travel is immaterial). See Figure 4.3*

Two edges of G are called siblings if neither lies in any cycle but they have a common vertex. Let t be a SAW that uses no pair of siblings. There exists no trail t' with the same endpoints as t that contains a copy (under $\text{Aut}(G)$) of $p(\square, v)$. On the

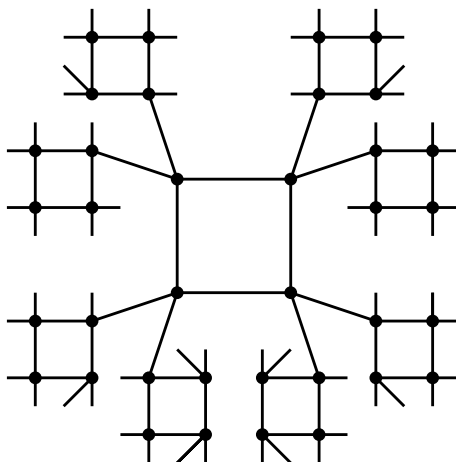


FIGURE 4.2. Part of the free product graph $G = \mathbb{Z}_4 * K_2 * K_2$. Any trail that circumnavigates the central square must pass along two siblings.

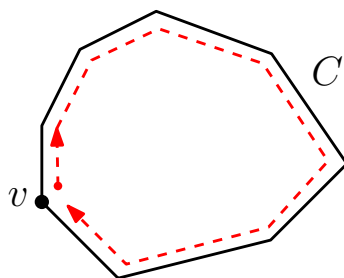


FIGURE 4.3. With C a cycle and $v \in C$, the t-pattern $p(C, v)$ is a circumnavigation of C starting and ending at v .

other hand there exists a trail containing three disjoint copies of $p(\square, v)$; cf. Remark 3.5.

It may be seen that G possesses a height function with respect to which $p(\square, v)$ is bridgeable.

5. PROOF OF THEOREM 4.5

Let $G \in \mathcal{G}$ have height function $(h, \mathcal{H})$, and let

$$(5.1) \quad d = d(h) = \max\{|h(u) - h(v)| : u, v \in V, u \sim v\}.$$

For clarity of exposition, we shall first assume that the height function is transitive (that is, $\mathcal{H}$ acts transitively on the vertex-set V), and then in Section 5.2 we explain the further steps in the non-transitive case. This mimics the approach of [15, 29].

5.1. Assume the height function is transitive. We assume in this subsection that the height function is transitive, and we recall from (4.5) that $\Pi = \emptyset$ in this case..

Lemma 5.1. *The pattern p satisfies $p \in \Sigma_b^\pm$ if and only if $p_{\text{rev}} \in \Sigma_b^\mp$.*

Proof. We may assume without loss of generality that $p \in \Sigma_b^+$, and we shall deduce that $p_{\text{rev}} = R(p)$ is $(-)$ bridgeable. Let $w = (w_0 = \mathbf{0}, w_1, \dots, w_n)$ be a $(+)$ bridge containing a copy of p . By adding a further edge if needed, we may assume that $h(w_n) > h(w_i)$ for $0 \leq i < n$. By the assumed transitivity of the height function, there exists $\alpha \in \mathcal{H}$ such that $\alpha(w_n) = \mathbf{0}$. Then $R(\alpha w)$ is a $(-)$ bridge containing p_{rev} , as required. $\square$

Proposition 5.2.

(a) *If $p \in \Sigma_b^\pm$, then*

$$(5.2) \quad \beta^\pm(p) < \beta^+.$$

(b) *If $p \in \Sigma_b$, then*

$$(5.3) \quad \mu(p \cup p_{\text{rev}}) < \mu(G).$$

Proof. We follow the proof of Proposition 3.7. The proof is given only in outline, and the reader is referred to [15, 29] for further details.

(a) Inequality (5.2) holds in the same manner as (3.12), and the details are omitted.

(b) For (5.3) a new argument is needed, and this is provided by the reversal approach of [15] as adapted in [29] and illustrated in Figures 5.1 and 5.2. We summarise first how the argument of [29] leads to the equality

$$(5.4) \quad \mu = \beta^+ \vee \beta^-$$

of Theorem 4.3. Evidently, $\mu \geq \beta^+ \vee \beta^-$, and it suffices for (5.4) to prove that

$$(5.5) \quad \mu \leq \beta^+ \vee \beta^-.$$

Let $v = (v_0, v_1, \dots, v_m)$ and $w = (w_0, w_1, \dots, w_n)$ be walks on G . Since $\mathcal{H}$ acts transitively, there exists $\alpha \in \mathcal{H}$ such that $\alpha(w_0) = v_m$. By $v \circ w$ we denote the concatenation of v followed by $(\alpha(w_1), \alpha(w_2), \dots, \alpha(w_n))$.

A *half-space walk* $w = (w_0, w_1, \dots, w_n)$ on G is a walk such $h(w_i) > h(w_0)$ for all $i \geq 1$. Let H_n denote the set of n -step half-space SAWs starting at $\mathbf{0}$, and let $w \in H_n$. The idea is to decompose w as an alternating sequence of $(+)$ bridges $x_1, x_3, \dots$ and $(-)$ bridges $x_2, x_4, \dots$, whose spans are decreasing, and to reposition these to obtain a $(+)$ bridge $y^+ := x_1 \circ x_3 \circ \dots$ and a $(-)$ bridge $y^- := x_2 \circ x_4 \circ \dots$. This is illustrated in Figure 5.1.

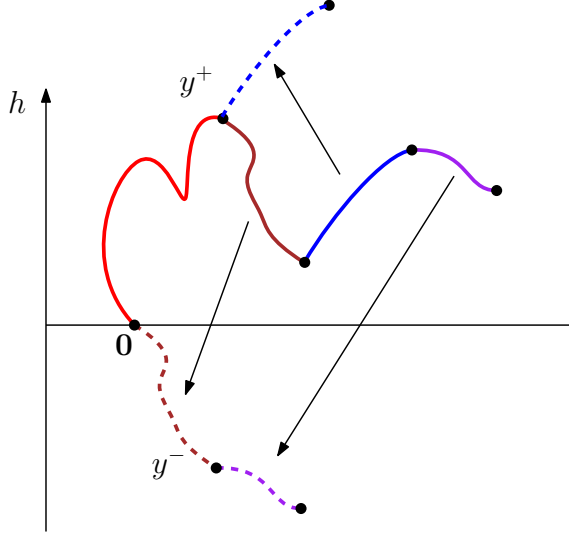


FIGURE 5.1. An illustration of the transformation from a half-space walk to a (+)bridge y^+ and a (-)bridge y^- .

Let a_i denote the span of x_i . Each such w is specified by knowledge of y^+ , y^- , and the sequence (a_i) . Let $h_n = |H_n|$ and let $h_n(a_1, a_2, \dots, a_k)$ be the number of elements of H_n with $a_i = \text{span}(x_i)$ for $1 \leq i \leq k$. As in [29, Lemma 3.3],

$$(5.6) \quad h_n = \sum_{k=1}^n \sum_{\substack{a_1 > a_2 > \dots > a_k > 0 \\ a_1 + a_2 + \dots + a_k \leq dn}} h_n(a_1, a_2, \dots, a_k).$$

On using the transformation illustrated in Figure 5.1,

$$(5.7) \quad \begin{aligned} h_n &\leq \sum_{k=1}^n \sum_{\substack{a_1 > a_2 > \dots > a_k > 0 \\ a_1 + a_2 + \dots + a_k \leq dn}} \sum_{m=0}^n b_m^+(a_1 + a_3 + \dots) b_{n-m}^-(a_2 + a_4 + \dots) \\ &\leq P(dn) \sum_{k=1}^n \sum_{m=0}^n b_m^+ b_{n-m}^-, \end{aligned}$$

where $b_s^\pm(r)$ denotes the number of s -step $(\pm)$ bridges with span r , and $P(K)$ is the number of (ordered) partitions of the integer K . Using the Hardy–Ramanujan partition theorem of [24], we deduce that there exist constants $A, B > 0$ such that, by (4.4),

$$(5.8) \quad h_n \leq e^{\pi\sqrt{dn/3}} \sum_{m=0}^n b_m^+ b_{n-m}^- \leq A e^{B\sqrt{n}} (\beta^+ \vee \beta^-)^n.$$

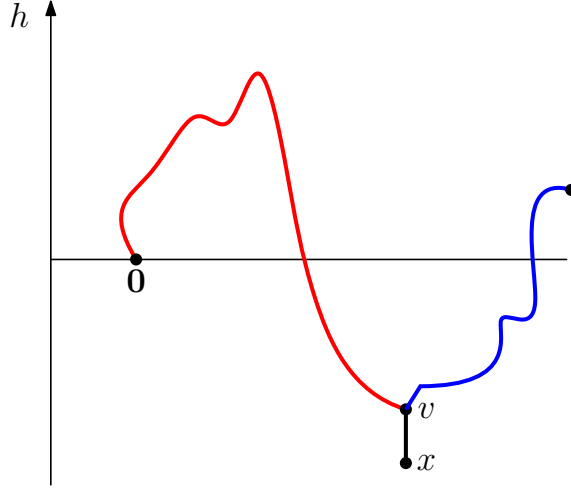


FIGURE 5.2. A SAW from $\mathbf{0}$ gives rise to two half-space walks starting at the additional vertex x . The leftmost such walk is a reversal of the original.

Next we relate half- and full-space SAWs, as in [23, 29], for example. See Figure 5.2. Let w be a SAW of G starting at $\mathbf{0}$, and find its last vertex v with minimal height. By breaking w at v into two sub-walks (one of which is reversed) and adding a new edge $\langle v, x \rangle$ to an additional vertex x with strictly smaller height than v , we find that

$$(5.9) \quad \sigma_n \leq \sum_{m=0}^n h_{n-m} h_{m+1},$$

whence (5.5) follows by (5.8). The equality of (5.4) follows.

We next repeat the above argument for walks containing copies of neither p nor p_{rev} . We may assume without loss of generality that $p \in \Sigma_{\text{b}}^+$; in the other case when $p \in \Sigma_{\text{b}}^-$, we recall Lemma 5.1 and work instead with p_{rev} . Inequalities (5.8) and (5.9) become, in the usual notation,

$$(5.10) \quad h_n(p \cup p_{\text{rev}}, 0) \leq e^{\pi\sqrt{dn/3}} \sum_{m=0}^n b_m^+(p \cup p_{\text{rev}}, 0) b_{n-m}^-(p \cup p_{\text{rev}}, 0) \\ \leq A e^{B\sqrt{n}} (\beta^+(p) \vee \beta^-(p_{\text{rev}}))^n,$$

$$(5.11) \quad \sigma_n(p \cup p_{\text{rev}}, 0) \leq \sum_{m=0}^n h_{n-m}(p \cup p_{\text{rev}}, 0) h_{m+1}(p \cup p_{\text{rev}}, 0).$$

Therefore, by (5.2), Lemma 5.1, and Theorem 4.3,

$$\begin{aligned}\mu(p \cup p_{\text{rev}}) &\leq \beta^+(p) \vee \beta^-(p_{\text{rev}}) \\ &< \beta^+ \vee \beta^- = \mu(G),\end{aligned}$$

as claimed in (5.3). $\square$

Part (a) of Theorem 4.5 is now proved, and part (b) follows as in the proof of Theorem 3.3, that is, by the argument of [33, Lemma 7.2.5].

5.2. Assume the height function is quasi-transitive. We assume in this subsection that $\mathcal{H}$ acts quasi-transitively but not transitively, and we shall outline the extra steps needed beyond the arguments of the last subsection. Further details may be found in the relevant parts of [15, 29].

The new difficulty and its solution are exemplified as follows. Let $v = (v_0, v_1, \dots, v_m)$ and $w = (w_0, w_1, \dots, w_n)$ be walks with $h(v_0) < h(v_m)$ and $h(w_0) < h(w_n)$, say. We may wish to extend v by appending a copy of w , while maintaining height differences in a natural way. In the transitive case, one may choose $\alpha \in \mathcal{H}$ such that $\alpha(w_0) = v_m$, and the extended path $v \circ w$ is obtained by adjoining v and $\alpha(w)$. This path has the property that

$$h(v_0) < h(v_m) = h(\alpha(w_0)) < h(\alpha(w_n)).$$

In the quasi-transitive case, such $\alpha \in \mathcal{H}$ may not exist. Instead, we find $\alpha \in \mathcal{H}$ such that $\alpha(w_0)$ is close to v_m , and in addition $h(\alpha(w_0)) > h(v_m)$; then we find a path from v_m to $\alpha(w_0)$ using only vertices with height strictly between $h(v_m)$ and $h(\alpha(w_0))$ (apart from its endvertices). Since V has only finitely many orbits under $\mathcal{H}$, the lengths of such paths may be taken to be uniformly bounded. This is formalized as follows.

Associated with a height function $(h, \mathcal{H})$ are two integers d, r . The constant d is as in (5.1). If $\mathcal{H}$ acts transitively, we set $r = 0$. Otherwise, let r be the infimum of all r such that the following holds. Let $o_1, o_2, \dots, o_M$ be representatives of the orbits of $\mathcal{H}$. For $i \neq j$, there exists $v_j \in \mathcal{H}o_j$ such that $h(o_i) < h(v_j)$, and a SAW $\nu(o_i, v_j)$ from o_i to v_j , with length r or less, all of whose vertices x , other than its endvertices, satisfy $h(o_i) < h(x) < h(v_j)$. We fix such a SAW, and denote it as above (with v_j its endvertex other than o_i); we set $\nu(o_i, o_i) = \{o_i\}$. Such paths $\nu(o_i, o_j)$, and their images under $\mathcal{H}$, are called *connectors*. It is proved in [15, Prop. 3.2] that $r < \infty$.

The idea now is to reproduce the geometrical constructions of Section 5.1 with the difference that, when re-positioning a sub-SAW, we may need to include a suitable connector near the join. Such connectors may themselves contain copies of a given pattern p (or its reversal p_{rev}), and this may disturb the overall pattern count. We choose therefore to exclude patterns that lie in any connector. The set of such patterns may be identified as follows.

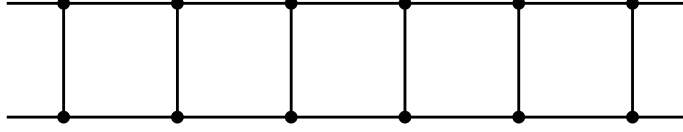


FIGURE 5.3. The ladder graph.

Let $G/\mathcal{H} = (\bar{V}, \bar{E})$ be the (simple) quotient graph with vertex-set the set of orbits of V under $\mathcal{H}$, and with edge-set as follows. For $v \in V$, write $\bar{v} = \mathcal{H}v$. For $v, w \in V$, we place an edge between $\bar{v}$ and $\bar{w}$ if and only if some member of $\bar{w}$ is joined by an edge of G to v (if $w \in \bar{v}$, this edge is a loop). See [12, Sect. 3.4] and [15, Sect. 2] for discussions of this quotient graph. Since $G/\mathcal{H}$ is connected, it has a spanning tree T . Let $L = L(T)$ be a lift of T into G , so that L is a sub-tree of G with M vertices. By the proof of [15, Prop. 3.2], we see that the above path $\nu(o_i, o_j)$, considered as a directed path from o_i to v_j , may be assumed to traverse edges of increasing height together with some SAW of L . It suffices therefore to exclude consideration of patterns that occur as such SAWs. Let Π denote the set of either (+)bridgeable or (−)bridgeable patterns that occur in G as SAWs of L , noting that

$$(5.12) \quad \text{every } p \in \Pi \text{ is a SAW,}$$

$$(5.13) \quad p \in \Pi \text{ if and only if } p_{\text{rev}} \in \Pi.$$

Example 5.3 (The ladder graph). *The ladder $G = \mathbb{L}$ of Figure 5.3 provides a simple illustration of the above discussion. Let $\mathbb{L} = (V, E)$ have vertex-set $V = \{0, 1\} \times \mathbb{Z}$ and edge-set the set of all pairs of vertices separated by unit distance. Clearly $\mathbb{L}$ is transitive, with automorphism group generated by the translation τ and reflection ρ given by*

$$\tau(v) = (v_1, v_2 + 1), \quad \rho(v) = (1 - v_1, v_2), \quad \text{for } v = (v_1, v_2) \in V.$$

Let $h(v) = v_2$ and let $\mathcal{H}$ be the infinite cyclic group generated by τ , so that $(h, \mathcal{H})$ is a quasi-transitive (but not transitive) height function. The lift L of the spanning tree of $G/\mathcal{H}$ may be taken to be the ‘vertical’ edge e between $(0, 0)$ and $(0, 1)$. Since e is not a cut-edge, we have $\Pi = \emptyset$

We introduce a slightly more general version of a bridge, to be called an ‘ebridge’. A $(\pm)$ ebridge is a SAW $w = (w_0, w_1, \dots, w_n)$ satisfying either (4.2) or (4.3) as appropriate; that is, we do not require ebridges to start at $\mathbf{0}$. An (i, j) -ebridge is a (+)ebridge that starts at o_i and ends in $\mathcal{H}o_j$. Let $b_n^{i,j}$ be the number of n -step (i, j) -ebridges, and $b_n^{i,\cdot} = \sum_j b_n^{i,j}$. An (i, i) -ebridge is called *i-prime* if it is minimal in the sense that it contains no strict-sub-SAW that is an (i, i) -ebridge.

Recall the number M of orbits of V under $\mathcal{H}$. As before, $b_n^\pm$ denotes the number of n -step $(\pm)$ bridges from $\mathbf{0}$.

Lemma 5.4.

- (a) The limit $\beta^+ = \lim_{n \rightarrow \infty} (b_n^+)^{1/n}$ exists, and $b_n^+ \leq (\beta^+)^{n+r}$ for $n \geq 1$.
- (b) For $i, j \in \{1, 2, \dots, M\}$, we have that $(b_n^{i,j})^{1/n} \rightarrow \beta^+$ as $n \rightarrow \infty$.
- (c) The pattern p satisfies $p \in \Sigma_b^+$ if and only if $p_{\text{rev}} \in \Sigma_b^-$.

Lemma 5.5. Let $p \in \Sigma_b^+$.

- (a) For $i \in \{1, 2, \dots, M\}$ and some n , there exists an n -step (i, i) -bridge containing a copy of p .
- (b) If, in addition, $p \notin \Pi$ then, for $i \in \{1, 2, \dots, M\}$,

$$\lim_{n \rightarrow \infty} b_n^{i,i}(p, 0)^{1/n} < \beta^+, \quad \lim_{n \rightarrow \infty} b_n^{i,\cdot}(p, 0)^{1/n} < \beta^+.$$

Hence, $\beta^+(p) < \beta^+(G)$.

Proof of Lemma 5.4. (a) Since p is $(+)$ -bridgeable, there exists some $(+)$ bridge w containing a copy of p . By adding appropriate connectors at the start and end of b , one obtains an (i, i) bridge containing a copy of p .

(b) This is similar to [15, Prop. 4.2]. Since $M < \infty$, there exist $a, b \in \{1, 2, \dots, M\}$ such that $\mathbf{0} \in \mathcal{H}o_a$ and $(b_n^{a,b})^{1/n} \rightarrow \beta^+$ as $n \rightarrow \infty$. Let $i, j \in \{1, 2, \dots, M\}$. By adding connectors at the ends of walks, there exists k satisfying $0 \leq k \leq 2r$ such that $b_n^{i,j} \geq b_{n-k}^{a,b}$ for $n \geq k$. The claim follows.

(c) This holds as in Lemma 5.1, with the same proof but subject to the addition of connectors where needed. $\square$

Proof of Lemma 5.5. (a) Since $p \in \Sigma_b^+$, there exist $a, b \in \{1, 2, \dots, M\}$ and an (a, b) -bridge w containing a copy of p . By adding suitable connectors at the ends of w , we obtain an (i, i) -bridge containing p .

(b) Let $p \in \Sigma_b^+ \setminus \Pi$ and let q_n^i be the number of n -step (i, i) -bridges that are i -prime. As in (3.12),

$$b_n^{i,i} = q_n^i + \sum_{m=1}^{n-1} q_m^i b_{n-m}^{i,i}, \quad n \geq 1.$$

Therefore, the generating functions

$$B^i(x) = \sum_{n=1}^{\infty} b_n^{i,i} x^n, \quad Q^i(x) = \sum_{n=1}^{\infty} q_n^i x^n,$$

satisfy

$$B^i(x) = \frac{Q^i(x)}{1 - Q^i(x)} \quad \text{whenever } Q^i(x) < 1.$$

By Lemma 5.4(b), B^i has radius of convergence $1/\beta^+$, whence $Q^i(1/\beta^+) = 1$.

Let Q_p^i be the generating function of i -prime (i, i) -bridges that contain no copy of p . Since p is assumed bridgeable, by part (a), $Q_p^i(1/\beta^+) < 1$. Since Q_p^i is continuous inside its ball of convergence, and $M < \infty$, there exists $\eta = \eta(p) > 0$ such that

$$(5.14) \quad Q_p^i((\beta^+)^{-1} + \eta) < 1 \quad \text{for all } i.$$

As following (3.17), we deduce that

$$\limsup_{n \rightarrow \infty} b_n^{i,i}(p, 0)^{1/n} \leq \frac{1}{(\beta^+)^{-1} + \eta} < \beta^+ \quad \text{for all } i.$$

The limit exists here by subadditivity. This proves the first inequality of part (b).

Finally, let $j \in \{1, 2, \dots, M\}$. By adding a copy of $\nu(o_i, o_j)$ at the end of an (i, i) -bridge w_n , we obtain an (i, j) -bridge w_{n+k} for some $k \leq r$, and moreover if w_n does not contain a copy of p , then neither does w_{n+k} (it is here that we use the assumption that $p \notin \Pi$). Therefore, $b_m^{i,j}(p, 0) \geq b_{m-k}^{i,i}(p, 0)$ for $m > k$. The claim follows on summing over j and taking the n th root. $\square$

Lemma 5.6. *Let $p \in \Sigma_b \setminus \Pi$. Then*

$$\mu(p \cup p_{\text{rev}}) \leq \beta^+(p) \vee \beta^-(p_{\text{rev}}), \quad \mu(G) = \beta^+ \vee \beta^-.$$

Once this is proved, the proof of the quasi-transitive case of Theorem 4.5 is completed as in the transitive case, by Lemma 5.5 followed by an appeal to the proof of [33, Lemma 7.2.5].

Proof of Lemma 5.6. We adapt the arguments of Section 5.1, noting that, since $p \notin \Pi$, no copy of either p or p_{rev} appears in any connector. Some further details follow.

In proving (5.8), it was explained how to turn a half-space walk w (starting at $\mathbf{0}$) into a $(+)$ bridge and a $(-)$ bridge. This was achieved by translations of $(+)$ sub-bridges and $(-)$ sub-bridges of w . In the non-transitive case, the endvertices of these sub-bridges do not generally coincide, and one needs to insert suitable connectors at the joins. This is explained in [15, 29] to which the reader is referred for further details (see also Figure 5.4). Equation (5.6) remains true, but with

$$\begin{aligned} h_n(a_1, a_2, \dots, a_k) &\leq (r+1)^{k-1} \sum_{m=0}^n \left(\sum_{s=0}^{kr} \sum_{t=0}^{\infty} b_{m+s}^+(a_1 + a_3 + \dots + t) \right) \\ &\quad \times \left(\sum_{s=0}^{kr} \sum_{t=0}^{\infty} b_{n-m+s}^-(a_2 + a_4 + \dots + t) \right) \\ &\leq (r+1)^k \sum_{m=0}^n \left(\sum_{s=0}^{kr} b_{m+s}^+ \sum_{s=0}^{kr} b_{n-m+s}^- \right). \end{aligned}$$

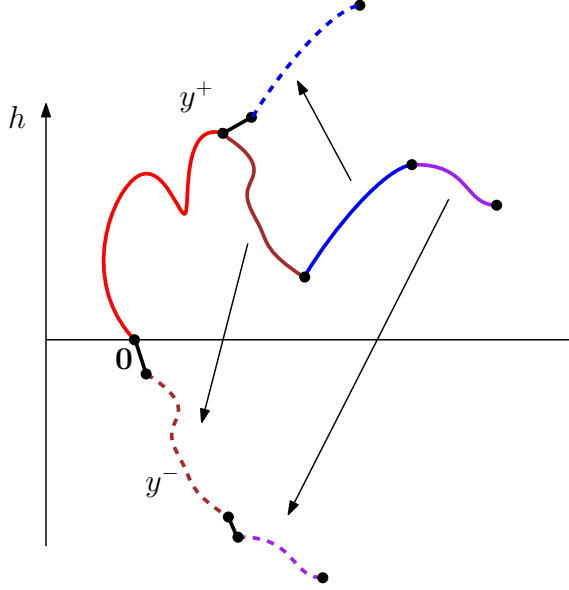


FIGURE 5.4. The transformation of Figure 5.1 with (black) connectors added where needed.

Therefore, as in [29, Lemma 3.7] and (5.7)–(5.8), there exists $B = B(G) < \infty$ such that

$$h_n \leq e^{B\sqrt{n}}(\beta^+ \vee \beta^-)^n.$$

This bound holds for half-space walks beginning at any given vertex, not only the root.

In a similar manner, (5.9) becomes

$$\begin{aligned} (5.15) \quad \sigma_n &\leq \sum_{m=0}^n e^{B\sqrt{n-m}} e^{B\sqrt{m+1}} (\beta^+ \vee \beta^-)^{n+1} \\ &\leq e^{C\sqrt{n}} (\beta^+ \vee \beta^-)^n \end{aligned}$$

for some $C = C(G) < \infty$. Therefore, $\mu(G) = \beta^+ \vee \beta^-$.

We may assume without loss of generality that $p \in \Sigma_b^+ \setminus \Pi$. We outline very briefly how to adapt the above sketch in order to bound the number of SAWs containing no copy of either p or p_{rev} . Corresponding to (5.15), we obtain on recalling (5.11) that

$$\sigma_n(p \cup p_{\text{rev}}, 0) \leq e^{C\sqrt{n}} [\beta^+(p) \vee \beta^-(p_{\text{rev}})]^n,$$

hence $\mu(p \cup p_{\text{rev}}) \leq \beta^+(p) \vee \beta^-(p_{\text{rev}}) < \mu(G)$, as required. $\square$

6. PROOF OF THEOREM 1.2

Let C be a cycle of $G \in \mathcal{G}$ and v a vertex of C . The t-pattern $p(C, v)$ is defined as a trail that starts and ends at v and traverses C (and no other edge) in between (see Figure 4.3). (Actually there are two such t-patterns depending on the direction of the traversal, but this will not be relevant in the following.) Since C is 2-edge-connected, $p(C, v)$ is indeed a t-pattern. We shall deduce Theorem 1.2 from the trail pattern theorem, Theorem 4.6, and the following two propositions.

Recall the set $\Sigma_{T,b}$ of t-bridgeable t-patterns.

Proposition 6.1. *Let $d \geq 4$, and let $G \in \mathcal{G}_d$ possess a transitive height function $(h, \mathcal{H})$. If G is not a tree, there exists a cycle C and a vertex v in C , such that $p(C, v) \in \Sigma_{T,b}$.*

Proposition 6.2. *Let $d \geq 4$. Suppose $G \in \mathcal{G}_d$ possesses some height function $(h, \mathcal{H})$, and there exists a cycle C of G and a vertex v of C such that the t-pattern $p(C, v)$ satisfies $p(C, v) \in \Sigma_{T,b}$. Then $\mu < \mu_T$.*

The following question is left open.

Question 6.3. *Does the conclusion of Proposition 6.1 still hold if one weakens the assumption on $(h, \mathcal{H})$ from transitivity to quasi-transitivity?*

An affirmative answer to this question would permit the removal of the assumption in Theorem 1.2 that the height function be transitive. A remark concerning the quasi-transitive case may be found in Example 4.7.

On considering graphs without height functions, it becomes natural to ask the following purely graph-theoretic question.

Question 6.4. *Let $G \in \mathcal{G}_d$ where $d \geq 4$, and suppose G is not a tree. Does there exist a cycle C and a vertex $v \in C$ such that v is 2-edge-connected to ∞ in the graph obtained from G after deletion of the edges of C ?*

The edge-connectivity of a subgraph of a *finite* graph, obtained by removing a path or cycle, has received some attention in the graph-theory literature, but Question 6.4 does not appear to have been answered. See [31, 37], and also [2, Lemma 3.3]. It is not hard to see that the answer is affirmative when G contains a 3-cycle.

Proof of Proposition 6.1. We shall show the existence of a cycle C such that the following holds. There exists a doubly-infinite trail $t = (\dots, t_{-1}, t_0, t_1, \dots)$ that includes a circumnavigation of C and satisfies

$$h(t_n) \rightarrow \begin{cases} -\infty & \text{as } n \rightarrow -\infty, \\ \infty & \text{as } n \rightarrow \infty. \end{cases}$$

By truncating t and adding connectors, we may obtain the required t-bridge containing $p(C, v)$.

Let D be a cycle of G . Let $v \in D$ be a vertex of D with maximal height, and let $w \in D$ have minimal height. Since G is transitive and $|D| \geq 2$, we may assume without loss of generality that

$$(6.1) \quad v \neq w, \quad h(w) \leq h(v).$$

By the definition of height function, there exists an infinite path $\pi_v = (v = v_0, v_1, \dots)$ of G such that $h(v_i)$ is strictly increasing in i ; similarly, there exists an infinite path $\nu_w = (w = w_0, w_1, \dots)$ of G such that $h(w_i)$ is strictly decreasing in i . We may choose π_v and ν_w such that

$$(6.2) \quad \begin{aligned} h(v_1) - h(v) &= \max\{h(v') - h(v) : v' \sim v\}, \\ h(w) - h(w_1) &= \max\{h(w) - h(w') : w' \sim w\}. \end{aligned}$$

Since $d \geq 4$, there exists an edge $\langle v, z \rangle$ such that z lies in neither the path π_z nor the cycle D . Let $\nu_z = (z = z_0, z_1, \dots)$ be an infinite path of vertices with strictly decreasing heights, and let z_r be the earliest vertex of π_z lying in D (if such a vertex does not exist, we set $r = \infty$). By (6.2), no vertex of π_z between $z = z_0$ and z_r lies in $\pi_v \setminus \{v\}$.

There are three cases.

- (a) Suppose $r = \infty$. The path ν_z , reversed, followed by the edge $\langle z, v \rangle$, followed by a traversal of D (in either direction), followed by π_v , contains a (+)t-bridge that traverses every edge of D . The proof is complete in this case (with $C := D$).
- (b) Suppose $r < \infty$ and $z_r \neq v$. Let z_s be the last vertex of ν_z lying in D . The cycle D is the union of two edge-disjoint sub-arcs a_1, a_2 from z_r to v , and we label these in such a way that $z_s \in a_1$. The sequence $v, z, z_1, z_2, \dots, z_r$, followed by the sub-arc a_2 , constitutes a cycle denoted C . The path $(z_s, z_{s+1}, \dots)$, reversed, followed by the sub-arc of a_1 from z_s to v , followed by a traversal (in either direction) of C , followed by π_v , contains a (+)t-bridge that traverses every edge of C . The proof is complete in this case.
- (c) Suppose $r < \infty$ and $z_r = v$. There are two sub-cases.
 - 1. If $r \geq 2$, we let C be the cycle $v, z, z_1, \dots, z_r = v$. The path ν_w , reversed, followed by a sub-arc of D from w to v , followed by a traversal of C , followed by π_v , contains a (+)t-bridge that traverses C . The proof is complete in this case.
 - 2. If $r = 1$, we have that $h(v) < h(z)$. The rest of this proof is devoted to this situation.

By the above, we may henceforth restrict our attention to the case when

$$(6.3) \quad \text{for all neighbours } z \text{ of } v \text{ not lying in } D \cup \pi_v, \text{ we have } h(v) < h(z).$$

By (6.3) and the maximality of $h(v)$ on D ,

$$(6.4) \quad v \text{ has exactly } d - 2 \text{ neighbours } z \text{ such that } h(v) < h(z).$$

By reversing the roles of v and w , we may assume that

$$(6.5) \quad w \text{ has exactly } d - 2 \text{ neighbours } z \text{ such that } h(w) > h(z),$$

in addition to (6.4).

Assume now that $(h, \mathcal{H})$ is transitive. By this transitivity, (6.4)–(6.5) hold for *all* vertices. This is a contradiction if $d \geq 5$, and the proof is complete in this case.

Assume $d = 4$. We shall use the notation (in particular, v, r) prior to case (a) above. By (6.5) and the transitivity of the height function, there are two possible choices for the first vertex z_1 of the path ν_z . We may therefore choose ν_z in such a way that $r \neq 1$. We now follow the analysis of cases (a)–(c) above to deduce the claim in this final case. $\square$

Only in the final two paragraphs of the last proof did we use the transitivity of the height function.

Proof of Proposition 6.2. Let the height function and the pair (C, v) be as given, and also the associated t-pattern $p(C, v)$. We may assume without loss of generality that $p(C, v) \in \Sigma_{T,b}^+$. By Theorem 4.6, all but an exponentially small proportion of n -step trails from $\mathbf{0}$ contain either p or p_{rev} . It follows that $\mu < \mu_T$. $\square$

Proof of Theorem 1.2. By Proposition 6.1, there exist C, v such that $p(C, v) \in \Sigma_{T,b}$. The required conclusion follows by Proposition 6.2. $\square$

7. SAWS ON LINE GRAPHS

The *line graph* (or *covering graph*) of a graph $G = (V, E)$ is the graph $G_K = (E, F)$ where F is the set of pairs of elements of E that share a vertex when viewed as edges of G . We picture the pair G, G_K as follows. The vertices of G_K are the midpoints of the edges of G , and two such vertices are connected in G_K by an edge whenever the corresponding edges of G share a vertex. Thus a vertex v of G with degree d is replaced by a copy, denoted $K(v)$, of the complete graph K_d on d vertices, having its vertices at the midpoints of edges of G incident with v . We shall refer to such copies of complete graphs as *kites* of G_K , and we write $\mathcal{K}$ for the set of such kites. The construction is illustrated in Figure 7.1 in the special case $G = \mathbb{Z}^2$.

Let $G \in \mathcal{G}$. Since G is transitive, G_K is quasi-transitive (as in the discussion following [17, Prop. 7.3]). Fix a root $\mathbf{0}$ for G , and a root $\mathbf{0}_K \in K(\mathbf{0})$ for G_K .

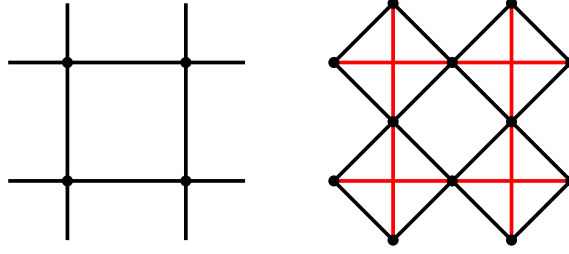


FIGURE 7.1. (L) Part of the square lattice $\mathbb{Z}^2$, and (R) its line graph $\mathbb{Z}_K^2$. Removal of the red edges inside the squares of $\mathbb{Z}_K^2$ results in a copy of $\mathbb{Z}^2$.

We introduce three sets of SAWs on G_K starting at $\mathbf{0}_K$. These sets differ in the manner in which they may behave on kites.

1. Σ is the set of all SAWs from $\mathbf{0}_K$ on G_K .
2. Σ^1 is the set of SAWs from $\mathbf{0}_K$ on G_K with the property that, each time they arrive in any given kite K , they must traverse exactly one edge of K before exiting K . They may return to K later.
3. Σ^- is the set of SAWs from $\mathbf{0}_K$ on G_K that may arrive (or begin) in any given kite at most once.

Write $\mu(G_K)$, $\mu^1(G_K)$, $\mu^-(G_K)$ for the respective exponential growth rates of n -step SAWs in these sets. These constants exist by arguments using subadditivity; see also [20].

A trail on G gives rise to a SAW on G_K . Moreover, the set of such trails generates the set Σ^1 , whence

$$(7.1) \quad \mu_T(G) = \mu^1(G_K).$$

We note a simple example of the use of a line graph.

Example 7.1. Let $G = \mathbb{Z}^2$, the square lattice, with line graph G_K as illustrated in Figure 7.1. The graph G_K contains a copy G' of G . By [12, Thm 3.2] with Γ the set of translations of G' , and $\mathcal{A}$ the subset of elements of Γ that preserve G_K , we have that $\mu(G') < \mu(G_K)$.

When restricted to cubic graphs (that is, every vertex-degree is 3) the above transformation is the so-called Fisher construction. The last was introduced by Michael Fisher [4] in his work on the Ising model, and it has been exploited in [11] in the context of SAWs (see also [7, 36]). It was shown in [11, Thm 1] that, when G is cubic and quasi-transitive, the connective constants of G and G_K are related by a simple formula. The Fisher construction was adapted in [17, Sect. 7] to transitive graphs

with degree 4, and the ensuing observations were used in [9, Thm 3.1] to find upper bounds for connective constants of Grigorchuk graphs.

Note that

$$(7.2) \quad \mu^-(G_K) = \mu(G_K) \quad \text{when } G \text{ is cubic.}$$

Theorem 7.2. *Let $G \in \mathcal{G}_d$ where $d \geq 3$. The connective constants $\mu(G)$ and $\mu^-(G_K)$ satisfy*

$$(7.3) \quad \mu(G_K) \geq \mu^-(G_K), \quad \text{and} \quad g_d\left(\frac{1}{\mu^-(G_K)}\right) = \frac{1}{\mu(G)},$$

where

$$(7.4) \quad g_d(\xi) = \sum_{l=1}^{d-1} \xi^l \frac{(d-2)!}{(d-l-1)!},$$

with the convention that $0! = 1$.

In the case $d = 3$, equation (7.4) differs from [11, Thm 1(a)] owing to the convention adopted here that the vertices of G_K lie at the midpoints of the corresponding edges of G .

Proof. The first inequality of (7.3) is an immediate consequence of the definitions. For the (second) equality, we follow the argument of [11] (see also the proof of [17, Prop. 7.4]), noting that there are $(d-2)(d-3) \cdots (d-l)$ paths of K_d of length l ($\leq d-1$) joining any two given distinct vertices of K_d . $\square$

The relevance of Theorem 7.2 to the analysis [9] of SAWs on Grigorchuk graphs is as follows. Let G_ω be the Cayley graph of the general Grigorchuk group Γ_ω corresponding to a given sequence $\omega \in \{0, 1, 2\}^\infty$ (see [9] for definitions). Let H denote the graph obtained from G_ω by contracting each of its kites (copies of K_4 in this case) to a single vertex; thus, $G_\omega = H_K$ with $d = 4$. In the proof of the main Theorem 3.1 of [9], a set W_n of n -step SAWs on G_ω is constructed that is based on the orbital Schreier graph $\mathcal{S}_\omega$. As noted in [9, Lemma 5.1], $W_n \subseteq \Sigma^-(G_\omega)$, whence its growth rate is bounded above by $\mu^-(G_\omega)$ which, in turn, is given by Theorem 7.2 in terms of $\mu(H)$.

DECLARATION

The author has no conflicts of interest, no data arose in this work, and no use was made of AI.

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