

CONNECTIVE CONSTANTS OF GRIGORCHUK GRAPHS

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ABSTRACT. The connective constant $\mu(G)$ of a graph G is the exponential growth rate of the number of self-avoiding walks starting at a given vertex. We prove upper and lower bounds for the connective constants of Cayley graphs G_ω of a general Grigorchuk group encoded by a sequence $\omega \in \{0, 1, 2\}^{\mathbb{N}}$. In particular, $\mu(G_\omega) > \phi$ for any such Cayley graph (subject to a simple condition on ω), where $\phi := \frac{1}{2}(1 + \sqrt{5})$ is the golden mean. This extends earlier work of the author and Zhongyang Li in “Cubic graphs and the golden mean”, Discrete Math. 343 (2020), article 111638, where it was conjectured that $\mu(G) \geq \phi$ for all infinite, vertex-transitive, cubic graphs. The current work includes an analysis of the proportions of appearances of given label-sequences in the orbital Schreier graphs of general Grigorchuk groups.

1. INTRODUCTION

Let G be an infinite, (vertex-)transitive, simple, rooted graph, and let σ_n be the number of n -step self-avoiding walks (SAWs) starting from the root. It was proved by Hammersley [?] in 1957 that the limit $\mu = \mu(G) := \lim_{n \rightarrow \infty} \sigma_n^{1/n}$ exists, and he called μ the ‘connective constant’ of G . A great deal of attention has been devoted to counting SAWs since that introductory mathematics paper, and survey accounts of many of the main features of the theory may be found at [?, ?, ?].

A graph is called *cubic* if every vertex has degree 3, and *transitive* if it is vertex-transitive. Let $\mathcal{G}_d$ be the set of connected, infinite, transitive, simple graphs with degree d , and let $\mu(G)$ denote the connective constant of $G \in \mathcal{G}_d$. The letter ϕ denotes the golden mean $\phi := \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \dots$. The following question was stated in [?] and investigated but not fully answered in [?].

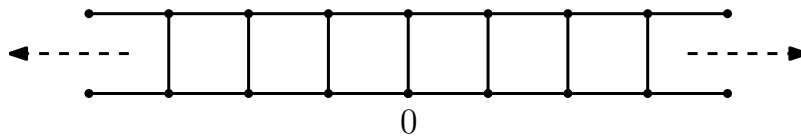
Question 1.1. *Is it the case that $\mu(G) \geq \phi$ for $G \in \mathcal{G}_3$?*

Our principal purpose is to answer this question affirmatively for the Cayley graphs of Grigorchuk groups, therein extending [?, Thm 8.1]. We state this loosely here,

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FIGURE 1.1. The ladder graph $\mathbb{L}$ with marked root.

with a mathematically precise, quantified, statement provided at Theorem 3.1. We recall that the general Grigorchuk group is encoded by an infinite sequence $\omega = (\omega_0, \omega_1, \omega_2, \dots) \in \{0, 1, 2\}^\infty$; see Section 2 for the details of this. Let Q be the set of all sequences ω that include all three values from the set $\{0, 1, 2\}$.

The Grigorchuk groups are usually defined with four generators, of which three (taken with the identity) form a Klein 4-group. Therefore, a Grigorchuk group has three distinct 3-generator Cayley graphs.

Theorem 1.2. *Let $\omega \in \{0, 1, 2\}^\infty$. Each Cayley graph CG_ω of a (3-generator) general Grigorchuk group Γ_ω satisfies $\mu(\text{CG}_\omega) \geq \phi$. Furthermore, the inequality is strict if $\omega \in Q$.*

See Theorem 3.1 for a fuller statement that includes rigorous upper and lower bounds for the connective constants in question.

Recall that the ‘ladder graph’ of Figure 1.1 has connective constant ϕ . One useful way to show that $G \in \mathcal{G}_3$ satisfies $\mu(G) \geq \phi$ is to find an injection from the set of SAWs on $\mathbb{L}$ into the set of SAWs on G . That is not the approach used here.

Several categories of cubic graphs are shown in [?] to satisfy $\mu \geq \phi$, but the general question remains open. One may consider its restriction to Cayley graphs of finitely generated groups (all Cayley graphs in this paper are in their *simple* form, that is, edges are undirected, and parallel edges are allowed to coalesce).

It is explained in [?] that the SAW analysis of such Cayley graphs varies depending on the number of ends of the group. Theorem 10.2 of [?] proves the required inequality for two-ended, cubic, Cayley graphs, and we note in passing the extension of this by Georgakopoulos and Wendland [?] to all two-ended, transitive, cubic graphs.

Theorem 1.3 ([?, Thm 1.3], [?, Thm 10.2]). *Let $d \geq 3$ and let $G \in \mathcal{G}_3$ be two-ended. Its connective constant satisfies $\mu(G) \geq \phi$.*

The proof, which is found in [?], proceeds as in [?] by constructing a non-constant, Lipschitz, ‘skew-difference-invariant’ harmonic function on G . The paper [?] contains also an example of a two-ended, transitive, cubic graph that is not a Cayley graph, thereby answering an old question of Watkins [?].

The Grigorchuk ‘branch groups’ form a category of groups that has attracted great interest, and this note is devoted to such groups. The work reported here extends

[?, Thm 8.1] which was concerned with the so-called first Grigorchuk group. In the context of self-avoiding walks, Grigorchuk groups are special in that they do not possess so-called graph height functions (see [?, Cor. 9.2] and [?, Thm C]).

We remind the reader of the use of generating functions in the context of SAWs. Let $\mathcal{W}$ be the set of SAWs from a given root in an infinite, transitive graph G , and let σ_n be the number of such n -step walks. The SAW generating function is given as

$$(1.1) \quad Z_G(\xi) = \sum_{w \in \mathcal{W}} \xi^{|w|} = \sum_n \sigma_n \xi^n, \quad \xi \in \mathbb{R},$$

where $|w|$ is the number of edges of w . The radius of convergence of Z_G is $1/\mu(G)$.

This paper is laid out as follows. Grigorchuk groups are defined in Section 2, the main theorem is presented in Section 3, and it is proved in Section 5.

One of the contributions of the current work is a direct derivation of the asymptotic proportions of appearances of given edge-label patterns in the orbital Schreier graphs of general Grigorchuk groups indexed by a vector ω . This is described in Section 4, and the relevant theorem is found at Theorem 4.1.

The set $\{0, 1, 2, \dots\}$ of non-negative integers is denoted $\mathbb{N}_0$. Given a finite string w , we write w^k (respectively, w^∞) for the string $ww \cdots w$ comprising k copies of w (respectively, a singly infinite string $ww \cdots$ of copies of w).

2. THE GRIGORCHUK GROUPS

2.1. The first Grigorchuk group. The (first) Grigorchuk group Γ was introduced in [?] (see also the more recent papers [?, ?, ?]) as a group of intermediate growth. It is defined as follows. Let T be the rooted binary tree with root labelled $\emptyset$. The vertex-set of T can be identified with the set of finite strings u having entries 0, 1, where the empty string corresponds to the root $\emptyset$, and the two children of a vertex labelled u are labelled $u0$ and $u1$. Let T_u denote the subtree of T with root u together with its ‘offspring’.

Let $\text{Aut}(T)$ be the automorphism group of T , and let $a \in \text{Aut}(T)$ be the automorphism that, for each string u , interchanges the two vertices $0u$ and $1u$.

Any $\gamma \in \text{Aut}(T)$ may be applied in a natural way to either subtree T_u , $u = 0, 1$. Given two elements $\gamma_0, \gamma_1 \in \text{Aut}(T)$, we define $\gamma = (\gamma_0, \gamma_1)$ to be the automorphism of T obtained by applying γ_0 to T_0 and γ_1 to T_1 . Define automorphisms b, c, d of T recursively as follows:

$$(2.1) \quad b = (a, c), \quad c = (a, d), \quad d = (\mathbf{1}, b),$$

where $\mathbf{1}$ is the identity automorphism.

The first Grigorchuk group Γ is defined as the subgroup of $\text{Aut}(T)$ generated by the set $S := \{a, b, c, d\}$. Since every generator has order 2, one may define its undirected

Cayley graph G with degree 4. We re-use the notation $\mathbf{1}$ for the identity element of Γ .

It is standard and easily checked that

$$(2.2) \quad b = cd = dc, \quad c = db = bd, \quad d = bc = cb.$$

Therefore, the set S is not a minimal generator set of Γ . More specifically, Γ is generated by any set of the form $S(\neg z) := S \setminus \{z\}$ for any given $z \in \{b, c, d\}$. We write $G(\neg z)$ for the (degree-3) Cayley graph of Γ with generator-set $S(\neg z)$.

The 3-neighbourhood of $\mathbf{1}$ in the Cayley graph $G(\neg d)$ of Γ is drawn in Figure 2.1 (subject to a relabelling of the vertices which will be clearer later in this work).

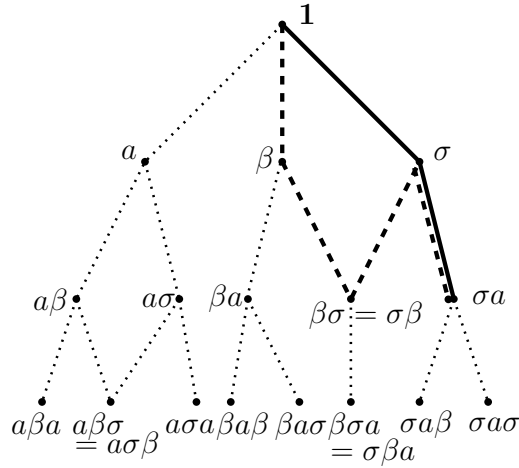


FIGURE 2.1. The 3-neighbourhood of the identity in $G_\omega(\neg d)$. Note that the walk $(\mathbf{1}, \sigma, \sigma a)$ may be re-routed as $(\mathbf{1}, \beta, \beta\sigma, \sigma, \sigma a)$.

2.2. General Grigorchuk groups. We turn now to the general Grigorchuk groups of [?] (see also [?, ?]), which are encoded as follows in terms of an infinite vector ω of 0s, 1s, and 2s. The first Grigorchuk group corresponds to the vector $012012012\dots$, abbreviated to $(012)^\infty$.

Let automorphisms $I(u)$ and $\Pi(u)$ act at vertices u of T in the following manner. The map $I(u)$ is the identity map that makes no change to T_u , whereas $\Pi(u)$ interchanges the two children $u0$ and $u1$ and their sub-trees.

Define the symbolic 3-vectors

$$(2.3) \quad \bar{0} = \begin{pmatrix} \Pi \\ \Pi \\ I \end{pmatrix}, \quad \bar{1} = \begin{pmatrix} \Pi \\ I \\ \Pi \end{pmatrix}, \quad \bar{2} = \begin{pmatrix} I \\ \Pi \\ \Pi \end{pmatrix}.$$

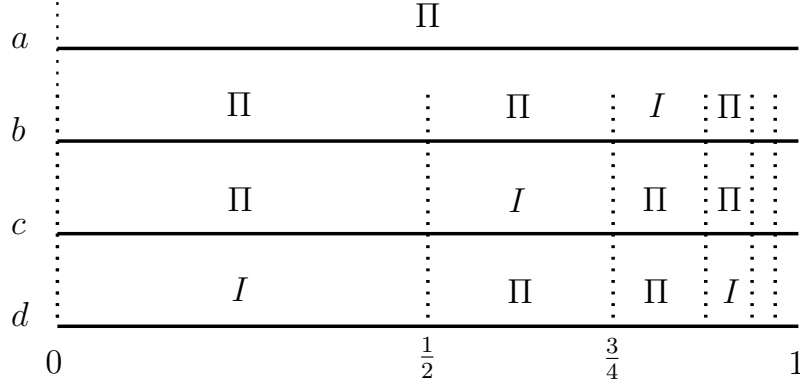


FIGURE 2.2. The unit interval $[0, 1]$ may be partitioned into the sub-intervals $[1 - 2^{-n}, 1 - 2^{-(n+1)})$ for $n \geq 0$. The generators a, b, c, d may be expressed as involutions of these sub-intervals. Each involution is either the identity I , or the permutation Π that interchanges the two halves of the corresponding sub-interval. Recalling (2.3)–(2.4), the illustrated example is encoded in the word $(012)^\infty$. The more general generator-set $\{a, b_\omega, c_\omega, d_\omega\}$ may be described similarly by adapting the columns to the elements of the vector ω .

Let $\omega = (\omega_n : n = 0, 1, 2, \dots) \in \Omega := \{0, 1, 2\}^{\mathbb{N}_0}$. (This indexing convention differs slightly from the usual one, but is convenient in the current setting.) Combine the column vectors $\overline{\omega_0}, \overline{\omega_1}, \dots$ into a $3 \times \infty$ matrix M and let $(b_n : n = 0, 1, 2, \dots)$ (respectively, $(c_n), (d_n)$) be the top row-vector (respectively, second, third row-vector) of M . We define four automorphisms $a, b_\omega, c_\omega, d_\omega$ by $a = \Pi(\emptyset)$, and

$$(2.4) \quad b_\omega = b_0(0) \circ b_1(10) \circ \dots \circ b_k(1^k 0) \circ \dots,$$

with a similar formula for c_ω (respectively, d_ω) with each b_n replaced by c_n (respectively, d_n). Here, $b_k(1^k 0)$ is an automorphism acting as b_n on the sub-tree $T_{1^k 0}$. In (2.4), for automorphisms α, β that act on disjoint vertex-subsets of T , the notation $\alpha \circ \beta$ means that they act simultaneously.

It is helpful to illustrate these generators as involutions of sub-intervals of $[0, 1]$, in the style of the initial papers in the field, namely [?, ?]. See Figure 2.2.

The general Grigorchuk group Γ_ω is defined as the group generated by the set $S_\omega := \{a, b_\omega, c_\omega, d_\omega\}$, with Cayley graph denoted G_ω . Since $\Pi^2 = I$, the elements of S_ω have order 2 and, since $\Pi^2 = I$ and $\Pi I = I \Pi = I$, we have that

$$(2.5) \quad b_\omega = c_\omega d_\omega = d_\omega c_\omega, \quad c_\omega = d_\omega b_\omega = b_\omega d_\omega, \quad d_\omega = b_\omega c_\omega = c_\omega b_\omega, \quad \text{for } \omega \in \Omega.$$

Therefore, S_ω is not a minimal generating set of Γ_ω , and one may discard any single element of $\{b_\omega, c_\omega, d_\omega\}$ to obtain an undirected, degree-3, Cayley graph denoted

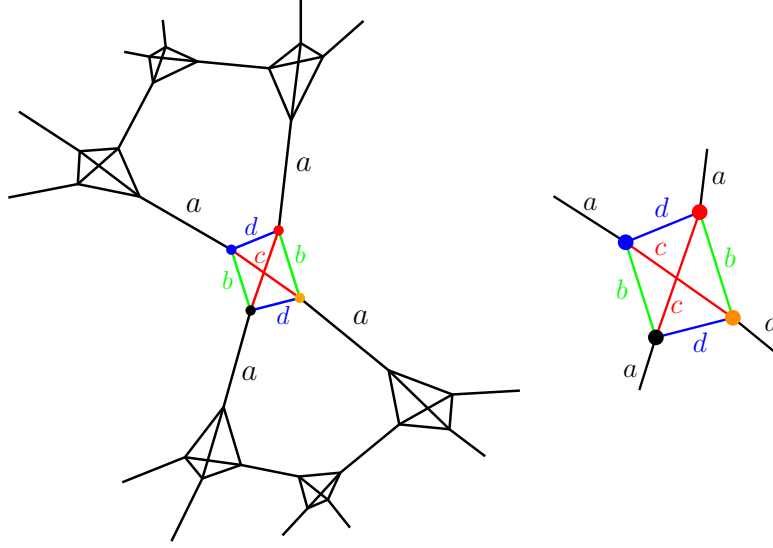


FIGURE 2.3. Part of the Cayley graph G of the first Grigorchuk group Γ . The Cayley graph G_ω of the general group Γ_ω is similar but the cycle structure may differ (though the kites remain). The letters b, c, d are then shorthand for $b_\omega, c_\omega, d_\omega$. *Left*: The black disk is the identity $\mathbf{1}$ and the central kite is the complete graph on the four vertices $\mathbf{1}, b, c, d$. *Right*: The central kite when expanded.

$G_\omega(\neg z)$ for Γ_ω , where z denotes the deleted generator. By (2.5), the set $\{\mathbf{1}, b_\omega, c_\omega, d_\omega\}$ forms a Klein 4-group.

The 4-generator Cayley graph G_ω of Γ_ω is sketched in Figure 2.3. A notable feature of this graph is the regular appearance of disjoint copies of the complete graph K_4 , each of which represents a translate of the above Klein 4-group. These subgraphs are termed *kites*, and we write $\mathcal{K}$ for the set of all kites.

3. MAIN THEOREM

Let $\omega \in \Omega$. Recall that $G_\omega \in \mathcal{G}_4$, and $G_\omega(\neg z) \in \mathcal{G}_3$ for $z \in \{b_\omega, c_\omega, d_\omega\}$.

Theorem 3.1. *Let $\omega \in \Omega$.*

- (a) *The connective constant $\mu(G_\omega)$ satisfies $\tau \leq \mu < \tau'$ where $\tau \approx 2.167$ is the reciprocal of the positive root of $2\xi^2(1 + 2\xi + 2\xi^2) - 1 = 0$, and $\tau' \approx 2.508$ is the reciprocal of the positive root of the equation*

$$(3.1) \quad B(\xi)(1 + D(\xi) + D(\xi)^2 + D(\xi)^3) - 1 = 0,$$

where $B(\xi) = 2D(\xi)$ and $D(\xi) = \xi^2(1 + 2\xi + 2\xi^2)$.

- (b) Let $\delta = \delta(\omega) \in \{b_\omega, c_\omega, d_\omega\}$ be such that $\delta_0 = I$. The connective constant $\mu(G_\omega(\neg\delta))$ satisfies $\gamma \leq \mu < \gamma'$, where $\gamma \approx 1.635$ is the reciprocal of the positive root of $2\xi^4(1 + \xi^2)(1 + \xi)^2 - 1 = 0$, and $\gamma' \approx 1.895$ is the reciprocal of the positive root of (3.1) with $B(\xi) = 2\xi^2(1 + \xi^2)$ and $D(\xi) = 2\xi^3$.
- (c) Let β, σ be the two elements of $\{b_\omega, c_\omega, d_\omega\}$ other than the δ of part (b), and let $z \in \{\beta, \sigma\}$. The connective constant $\mu(G_\omega(\neg z))$ satisfies $\phi \leq \mu < \gamma''$, where ϕ is the golden mean and $\gamma'' \approx 1.892$ is the reciprocal of the positive root of (3.1) with $B(\xi) = \xi^2(1 + \xi)^2$ and $D(\xi) = \xi^2(1 + \xi^2)$. Furthermore, the strict inequality $\mu(G_\omega(\neg z)) > \phi$ holds if ω is such that there exists $k \geq 1$ with $z_k = z_k(\omega) = I$.

The proof is found in Section 5. The proof of part (c) makes use of the results of Section 4 concerning the densities of label-sequences in the orbital Schreier graph of the ray 1^∞ of the tree T .

The bounds of Theorem 3.1 provide some information about the numerical values of the connective constants, and they may presumably be improved by using more intricate combinatorics. Rudi Mrazović [?] has kindly pointed out some numerical improvements to the bounds reported in an earlier version of this paper.

Remark 3.2. When $\omega = (012)^\infty$, Γ_ω is the first Grigorchuk group Γ , in which case $b_\omega = b = \beta$, $c_\omega = c = \sigma$, and $d_\omega = d = \delta$. A weaker version of part (b) was proved in [?, Thm 8.1] in this special case, namely that $\mu(G_\omega(\neg d)) \geq \phi$. The proof in Section 5 of the lower bound of Theorem 3.1(b) is a slightly streamlined version of that used in [?].

Remark 3.3. Note that the (weak) numerical bounds in Theorem 3.1(b, c) depend primarily on the value of ω_0 .

Remark 3.4. The sets of SAWs used in the proofs of the lower bounds of Theorem 3.1 are restricted to walks w with two further properties: (i) w does not arrive twice (or more) in any given kite, and (ii) w may be extended to an infinite SAW with the foregoing property. (Infinitely extendable SAWs were studied in [?].) The final strict inequality of part (c) may be made explicit by exploring the positive root of the forthcoming equation (5.8).

Remark 3.5. Muchnik and Pak [?] considered the critical value $p_c(G_\omega)$ of bond percolation on G_ω , and showed that $p_c(G_\omega) < 1$ for all ω . Such strict inequality has been extended to graphs with super-linear growth by Duminil-Copin et al., [?, Thm 1.3]. (The reader is referred to [?] for an introduction to percolation theory.) The upper bounds of Theorem 3.1 provide lower bounds for the critical values $p_c(G_\omega)$ via the well-known inequality $p_c(H) \geq 1/\mu(H)$ for any connected, infinite, transitive graph H (see the proof of [?, eq. 1.13]).

As remarked above, the first Grigorchuk group Γ may be identified as Γ_ω with $\omega = (012)^\infty$. Its orbital Schreier graph $\mathcal{S}$ (see Figure 4.1) is a map of the action of the generator-set of Γ on the ray 1^∞ . Its leftmost vertex is 1^∞ , and the progressive application of generators results in the sequence of rays given on the left side of Table 4.1.

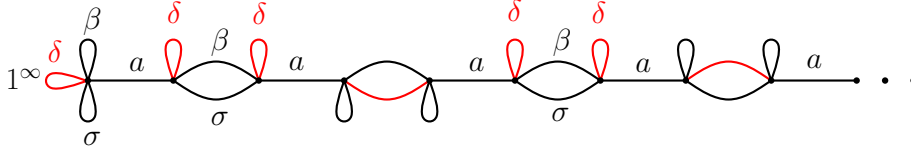


FIGURE 4.2. The orbital Schreier graph $\mathcal{S}_\omega$ of the ray 1^∞ for the Grigorchuk group Γ_ω with generator-set $\{a, \beta, \sigma, \delta\}$. The automorphisms β, σ are the two generators with first entry Π , and δ is the remaining generator. Between consecutive pairs of a , there are two loops and two non-loops. Between the k th and $(k+1)$ th appearances of a (for k odd), these edges are as labelled, whereas when k is even these edge-labels depend on ω and the edges' positions in $\mathcal{S}_\omega$. Deletion of the red edges gives rise to the Schreier graph $\mathcal{S}_\omega(-\delta)$ associated with the generator-set $\{b_\omega, c_\omega, d_\omega\} \setminus \{\delta\}$.

Let $\omega \in \Omega$. Let $\delta = \delta(\omega) \in \{b_\omega, c_\omega, d_\omega\}$ be the unique element such that $\delta_0 = I$, and let β, σ denote the other two generators. According to (2.3), each $k \in \{0, 1, 2\}$ corresponds to a 3-vector $\bar{k}$, and we let λ_k be the generator whose element in $\bar{k}$ is I , and write $\{\beta, \sigma, \delta\} \setminus \{\lambda_k\} = \{\mu_k, \tau_k\}$. Thus,

$$(4.1) \quad \lambda_0 = \delta, \quad \{\lambda_1, \lambda_2\} = \{\beta, \sigma\}.$$

The Schreier graph $\mathcal{S}_\omega$, illustrated in Figure 4.2, is graph-isomorphic to $\mathcal{S}$ but is labelled differently. The corresponding sequence of rays is identical to that of $\mathcal{S}$, that is, as given in Table 4.1. The labels of $\mathcal{S}_\omega$ are as in Figure 4.2, where two parallel unlabelled edges have distinct labels from $\{\beta, \sigma, \delta\}$ and the incident loops have the third label. The missing labels are assigned as follows.

Let $v_1, v_2, \dots$ be the vertices of $\mathcal{S}_\omega$ from left to right. Let $m \geq 1$ and consider v_{4m} and the corresponding ray ρ_{4m} of T (see Table 4.1). The first letter of ρ_{4m} is 1, and there follows a sequence of 1s followed by 0; thus, we write $\rho_{4m} = 1^a 0x$ where $a = a_m \geq 1$ and $x = x_m \in \{0, 1\}^\infty$. On following the prescription of Section 2, we see that the loop of $\mathcal{S}_\omega$ at v_{4m} is labelled λ_{ω_a} , and the two rightward edges at v_{4m} are labelled μ_{ω_a} and τ_{ω_a} , respectively. The above construction may be described in diagrammatic form as

$$(4.2) \quad v_{4m} \longrightarrow \rho_{4m} \longrightarrow 1^a 0x \longrightarrow \omega_a \longrightarrow \lambda_{\omega_a}.$$

For given $\ell \in \{\beta, \sigma, \delta\}$, how frequently does ℓ appear as the label of the loop at some v_{4m} ? The answer is implied by the next theorem. The indicator function of A is written $1(A)$. The set of finite words w in the alphabet $\{0, 1\}$ is denoted $\{0, 1\}^*$; the length of $w \in \{0, 1\}^*$ is written $|w|$, and $\emptyset$ denotes the empty word. The word

TABLE 4.1. The rays of T corresponding to the first 25 vertices of the Schreier graphs $\mathcal{S}_\omega$ with (*left*) $\omega = (012)^\infty$ and (*right*) $\omega = (02112000\cdots)$. When applied at any given vertex v_k with $k \neq 1$, one generator creates a loop, and the other two lead to the next vertex v_{k+1} . The labels in the two columns differ from each other in the blue rows, that is, at each v_{4m} for $m \geq 1$ (they also differ at each v_{4m+1} , but that will not concern us).

First Grigorchuk group, $\omega = (012)^\mathbb{N}$				$\omega = (02112000\cdots)$	
vertex v_k	ray ρ_k	loop	next labels	loop	next labels
1	1^∞	b, c, d	a	b, c, d	a
2	01^∞	d	b, c	d	b, c
3	001^∞	d	a	d	a
4	101^∞	c	b, d	b	c, d
5	1001^∞	c	a	b	a
6	0001^∞	d	b, c	d	b, c
7	0101^∞	d	a	d	a
8	1101^∞	b	c, d	c	b, d
9	11001^∞	b	a	c	a
10	01001^∞	d	b, c	d	b, c
11	00001^∞	d	a	d	a
12	10001^∞	c	b, d	b	c, d
13	10101^∞	c	a	b	a
14	00101^∞	d	b, c	d	b, c
15	01101^∞	d	a	d	a
16	11101^∞	d	b, c	c	b, d
17	111001^∞	d	a	c	a
18	011001^∞	d	b, c	d	b, c
19	001001^∞	d	a	d	a
20	101001^∞	c	b, d	b	c, d
21	100001^∞	c	a	b	a
22	000001^∞	d	b, c	d	b, c
23	010001^∞	d	a	d	a
24	110001^∞	b	c, d	c	b, d
25	110101^∞	b	a	c	a

$w \in \{0, 1\}^*$ is called *periodic with period p* if there exists $i = i(w)$ such that ρ_{4m} begins with the sequence $1w$ if and only if $m = i + rp$ for some $r \geq 1$.

Theorem 4.1. *Each word $w \in \{0, 1\}^*$ is periodic with period $2^{|w|}$. In particular, the limit*

$$\pi_w := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n 1(\rho_{4m} \text{ begins } 1w)$$

exists and satisfies

$$(4.3) \quad \pi_w = \frac{1}{2^{|w|}}.$$

Theorem 4.1 may be expressed as saying that the empirical distribution of the rays (ρ_{4m}) converges weakly to the uniform distribution on the part of the boundary ∂T that begins 1. By consideration of the actions of the generators of Γ_ω on the rays, this weak limit extends to the full sequence (ρ_k) with the limit being uniform on ∂T .

Corollary 4.2. *Let $\ell \in \{\beta, \sigma, \delta\}$. The limit*

$$\pi_\ell := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n 1(\text{the loop at } v_{4m} \text{ is labelled } \ell)$$

exists and satisfies

$$\pi_\ell = \sum_{a=1}^{\infty} \frac{1}{2^a} 1(\lambda_{\omega_a} = \ell).$$

If, for example, ω is a vector of independent random variables, each being uniformly distributed on $\{0, 1, 2\}$, the proportion of appearances of the loop-label ℓ may be written as $Z = \sum_{a=1}^{\infty} X_a / 2^a$ where the X_a are independent with the Bernoulli distribution with parameter $\frac{1}{3}$.

Remark 4.3. *It is likely that the above and perhaps more may be (or have been) obtained using the theory of symbolic dynamics developed in [?, ?, ?, ?] and elsewhere, but we prefer to proceed directly here.*

Proof of Theorem 4.1. The mapping from the vertex-set of $\mathcal{S}_\omega$ into the set ∂T of infinite rays is an injection (see, for example, [?, Thm 7.3], [?, p. 29]) and is illustrated in Table 4.1. Note that this mapping is independent of the choice of ω . The initial ray (that is, the image of the root of $\mathcal{S}_\omega$) is the ray $\rho_1 = 1^\infty$. We shall consider the evolution of the sequence (ρ_k) of rays as k grows. It is convenient to concentrate on suffices k that are multiples of 4, and we call the evolution from any ρ_{4m} to $\rho_{4(m+1)}$ a *stage* of the process.

It is immediate that the first letter of every ρ_{4m} is 1. Next we perform an illustration of the required argument.

Suppose m is such that $v_{4m} = 11x$ for some $x \in \{0, 1\}^\infty$, $x \neq 1^\infty$. The four moves that constitute the next stage are as follows.

1. The first letter following the first 0 is flipped, resulting in $11x'$ for appropriate x' .
2. The first letter of the ensuing word is flipped, giving $01x'$.
3. The second letter of the ensuing word is flipped, giving $00x'$.
4. The first letter of the ensuing word is flipped, giving $10x'$.

This sequence results in the word $\rho_{4m+4} = 10x'$. We express this by writing

$$(4.4) \quad 11 \longrightarrow 10.$$

On applying the same argument to the word $10y$ with $y \in \{0, 1\}^\infty$, we find that $10 \longrightarrow 11$, and thus we may write

$$(4.5) \quad 11 \longrightarrow 10 \longrightarrow 11.$$

By (4.4)–(4.5), the words 0 and 1 are periodic with period 2. Hence, $\pi_0 = \pi_1 = \frac{1}{2}$ in agreement with the claim of the theorem.

The corresponding relations with three letters are

$$(4.6) \quad 111 \longrightarrow 101 \longrightarrow 110 \longrightarrow 100 \longrightarrow 111,$$

so that the limits $\pi_{11}, \pi_{01}, \pi_{10}, \pi_{00}$ exist and each equals $\frac{1}{4}$.

It is instructive to combine (4.5) and (4.6) in a vertical table. Table 4.2 illustrates the nested structure of the relations (4.5)–(4.6). Some comments follow.

- (a) Let $a \geq 1$, and suppose we have tabulated the evolution of the initial $(a + 1)$ -subsequences of the rays, starting at 1^{a+1} , and continuing until the first reappearance of 1^{a+1} .
- (b) Let A_a be the statement that, for every $w \neq 1^a$ with $|w| = a$, the word $1w$ appears in this tabulation exactly once. (The word $1 \cdot 1^a$ appears twice, once at each end.) Table 4.2 includes such tabulations for $a = 1, 2$, and the assumption is satisfied for these values..
- (c) *Assume A_a holds.* Take two copies of this tabulation, the second of which is appended beneath the first. We add the appropriate $(a + 2)$ th column, noting that it starts with its letter 1, and that that it changes its value each time it passes the word $1^a 0$ in the left $a + 1$ columns, and not elsewhere.
- (d) The word $1^a 0$ occurs exactly once in each of the two tabulations, and at the same position.
- (e) It follows by A_a that A_{a+1} holds. Since A_1 holds, every A_a holds by induction.

In conclusion, for every $a \geq 1$, the evolution beginning 1^{a+1} , until its first reappearance, includes every $1w$ with $|w| = a$, $w \neq 1^a$, exactly once. That is to say, the evolution

$$(4.7) \quad 1^{a+1} \longrightarrow 101^{a-1} \longrightarrow 1101^{a-2} \longrightarrow 10^3 1^{a-3} \longrightarrow \dots \longrightarrow 1001^{a-2} \longrightarrow 1^{a+1},$$

TABLE 4.2. *Left*: Two successive copies of the display (4.5) are displayed vertically on the left side, followed in the third column by the extensions given in (4.6). As one descends the third column, there are exactly two changes in value, at the cells coloured blue, each corresponding to the previous word 10 (coloured pink) in the two left columns. *Right*: Two copies of (4.6) displayed vertically. The two changes in the fourth column correspond to the occurrences of 110 in the previous row.

				1	1	1	1
				1	0	1	1
				1	1	0	1
				1	0	0	0
				1	1	1	0
				1	0	1	0
				1	1	0	0
				1	0	0	1
				1	1	1	1

includes a unique copy of any word of the form $1w$ with $|w| = a$, $w \neq 1^a$. Therefore, such w has period $2^{|a|}$, whence the limit π_w exists for all w and satisfies (4.3). $\square$

Proof of Corollary 4.2. Let $a \geq 1$, $\ell \in \{\beta, \sigma, \delta\}$, and let $h_m(a, \ell)$ be the indicator function that $a_m = a$ and in addition $\lambda_{\omega_a} = \ell$. (Recall (4.1) and the notation leading to (4.2).) Then,

$$\pi_\ell = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \sum_{a=1}^{\infty} h_m(a, \ell),$$

and the claim holds by Theorem 4.1 on interchanging the summations. $\square$

5. PROOF OF THEOREM 3.1

The proof is basically the same whatever the choice of ω , and therefore the reader loses little by restricting attention to the case $\omega = (012)^\infty$ and $b_\omega = b = \beta$, $c_\omega = c = \sigma$, $d_\omega = d = \delta$. For completeness we shall include the details even in places where they resemble the proof of [?, Thm 3.1].

Throughout this proof we assume $\omega \in \Omega$, and we let β, σ, δ be given as in the statement of the theorem, that is, $\delta = \delta(\omega) \in \{b_\omega, c_\omega, d_\omega\}$ is such that $\delta_0 = I$, and β, σ are the two remaining elements of $\{b_\omega, c_\omega, d_\omega\}$.

5.1. Case (a). Let $\mathcal{W}_0$ be the set of finite, labelled walks on the Schreier graph $\mathcal{S}_\omega$ starting at the root 1^∞ that, at each step, either move one step rightwards or pass around a loop (no loop may be traversed more than once). Members of $\mathcal{W}_0$ may be considered as finite words in the alphabet $\{a, \beta, \sigma, \delta\}$ without consecutive repetitions. Walks in $\mathcal{W}_0$ generally contain loops and are therefore not self-avoiding on $\mathcal{S}_\omega$, but we shall see next that they give rise to a certain set $\overline{\mathcal{W}_0}$ of self-avoiding walks on G_ω starting at its root $\mathbf{1}$.

Let $\mathcal{W}_0(\infty)$ be the set of *infinite* words constructed from $\mathcal{S}_\omega$ subject to the rules used in defining $\mathcal{W}_0$, and let $\overline{\mathcal{W}_0}(\infty)$ be the set of walks on G_ω obtained as lifts of elements of $\mathcal{W}_0(\infty)$.

Recall the set $\mathcal{K}$ of kites. Let $w = w_0 w_1 \cdots \in \mathcal{W}_0(\infty)$ and $\kappa \in \mathcal{K}$. Write $w^n = w_0 w_1 \cdots w_n$, a walk/word on $\mathcal{S}_\omega$ that lifts to the walk $\overline{w^n}$ on G_ω . We say that $\overline{w}$ *enters* κ at epoch n if either $n = 0$ and $\overline{w^0} \in \kappa$, or $n \geq 1$, $\overline{w^n} \in \kappa$, $\overline{w^{n-1}} \notin \kappa$.

Lemma 5.1.

- (a) Every $w \in \mathcal{W}_0(\infty)$ lifts to an infinite SAW $\overline{w}$ in G_ω from the root $\mathbf{1}$.
- (b) For $w \in \mathcal{W}_0(\infty)$ and $\kappa \in \mathcal{K}$, the infinite SAW $\overline{w} = (w_1, w_2, \dots)$ does not enter κ more than once.
- (c) Every $\overline{w} \in \overline{\mathcal{W}_0}$ is the initial sub-walk of an infinite SAW lying in $\overline{\mathcal{W}_0}(\infty)$.

Proof. (a) Let $w \in \mathcal{W}_0(\infty)$ and suppose $\overline{w}$ is not a SAW. Then w contains some shortest subword s of length 3 or more satisfying $s = \mathbf{1}$. On considering the action of Γ_ω on $\mathcal{S}_\omega$ (see Figure 4.2), we deduce that $\mathcal{S}_\omega$ contains a cycle of length 3 or more. By inspection of $\mathcal{S}_\omega$, this is seen to be a contradiction.

(b) Suppose $\overline{w} \in \overline{\mathcal{W}_0}(\infty)$ enters some $\kappa \in \mathcal{K}$ twice (or more often). In the order of traversal of $\overline{w}$, write v_1 for the first departure point from κ , and v_2 for the first subsequent arrival point in κ . Then v_1, v_2 are at least distance 4 apart in $\overline{w}$. Since $v_1, v_2 \in \kappa$, there exists a generator $z (\neq a)$ such that $v_2 = v_1 z$, whence $\mathcal{S}_\omega$ contains some cycle of length 4 or more, a contradiction.

(c) This holds since every word in $\mathcal{W}_0$ may be extended to an infinite word in $\mathcal{W}_0(\infty)$. $\square$

Let $R = \{r_1, r_2, \dots\}$ be the set of right-hand endpoints of the a -edges of $\mathcal{S}_\omega$ labelled in the order they are encountered when moving to the right from 1^∞ (in the sense of Figure 4.2). An ordered pair of elements $r_i, r_j \in R$ is called *consecutive* if $|i - j| = 1$. Let $\mathcal{W}$ be the subset of $\mathcal{W}_0$ containing words that end in a . As above, $\mathcal{W}$ lifts to a set $\overline{\mathcal{W}}$ of SAWs on G_ω .

We think of the set R as being points of renewal of walks in $\mathcal{W}$. More specifically, each $w \in \mathcal{W}$ can be broken into sections (called *units*) beginning and ending (respectively) with a consecutive pair $z, z' \in R$, and each unit σ has the form of Figure 5.1.

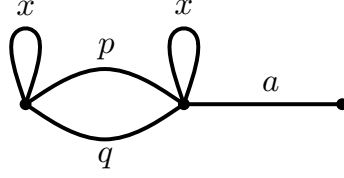


FIGURE 5.1. A unit of the Schreier graph $\mathcal{S}_\omega$ with general labels. The counts (according to length) of SAWs from the left end to the right end of the unit have generating function $A(\xi) = 2\xi^2 + 4\xi^3 + 2\xi^4$.

The generating function

$$(5.1) \quad Z(\xi) = \sum_{w \in \overline{\mathcal{W}}} \xi^{|w|}$$

of $\overline{\mathcal{W}}$ may be expressed in the form

$$(5.2) \quad Z(\xi) = A_0(\xi) \sum_{n=0}^{\infty} A(\xi)^n,$$

where $A_0(\xi)$ is the generating function of walks from 1^∞ to r_1 , and A is given in the caption of Figure 5.1. Since $Z(\xi) \leq Z_{G_\omega}(\xi)$ (for $\xi \geq 0$) and Z_{G_ω} has radius of convergence $1/\mu(G_\omega)$ (recall (1.1)), the radius of convergence of Z is at least $1/\mu(G_\omega)$, whence $\mu(G_\omega) \geq \zeta$ where $1/\zeta$ is the positive root of the equation $A(\xi) = 1$. This provides a lower bound for $\mu(G_\omega)$ that may be improved somewhat by the following variant of the argument used in [?, Sect. 8].

By Lemma 5.1, each SAW $\overline{w} \in \overline{\mathcal{W}}_0$ visits no kite κ more than once, and its route through such κ can be any of the eight words implied by the labelling of the associated unit of $\mathcal{S}_\omega$. Using Figure 5.1 for illustration, these words are pa , qa , xpa , xqa , pxa , qxa , $xpxa$, $xqxa$. To these we may add the two words $pqpa$, $qpqa$; see Figure 2.3 for an illustration of the two corresponding SAWs within κ . The above function $A(\xi)$ is now replaced by $\hat{A}(\xi) = 2\xi^2 + 4\xi^3 + 4\xi^4$, whence $\mu(G_\omega)$ is bounded below by the reciprocal of the positive root of $\hat{A}(\xi) = 1$.

Turning to the upper bound on $\mu(G_\omega)$, one may use the fact that G_ω has degree 4 and girth 3, and apply [?, Thm 7.3]. An improved upper bound is obtained by considering the Cayley graph G' of the group Γ' obtained from Γ_ω by removing all relators except (2.2). This G' is isomorphic to the free product graph $K_2 * K_4$ which, by [?, Thm 3.3], has connective constant μ' given as the positive root of $\xi^4 - 3\xi^2 - 6\xi - 6 = 0$. (Here, K_m denotes the complete graph on m vertices.) By [?, Cor. 4.1], we have the strict inequality $\mu(G_\omega) < \mu'$.

A slightly improved upper bound holds by an approach proposed by Rudi Mrazović [?], as follows. Let $\delta = \delta(\omega) \in \{b_\omega, c_\omega, d_\omega\}$ be such that $\delta_0 = I$, and let β and σ be

the generators with $\beta_0 = \sigma_0 = \Pi$. Note that $(a\delta)^4 = \mathbf{1}$. Let H be the graph obtained from G_ω by contracting every kite to a point (recall Figure 2.3). Since $(a\delta)^4 = \mathbf{1}$, every vertex of H lies in a cycle of length 4; such a cycle is called δ -cycle.

We may think of a walk w on G_ω as a walk w_H on H combined with its sub-walks within the kites encountered. As w passes through a kite κ , we call κ a δ -kite if the entry vertex e (respectively, departure vertex f) of w at κ satisfy $f = e\delta$. Note that every SAW w of G_ω has the property that it has no sequence of 4 consecutive δ -kites (since otherwise w_H would contain a δ -cycle).

Let $\mathcal{V} = \bigcup_n \mathcal{V}_n$ where $\mathcal{V}_n$ is the set of n -step walks $v = (v_0, v_1, \dots, v_n)$ on G_ω such that (i) $v_0 = \mathbf{1}$, $v_1 \in \{\beta, \sigma, \delta\}$, (ii) v is a SAW within each kite encountered, (iii) v_n is the entry vertex of some kite, and (iv) v has no subsequence of 4 consecutive δ -kites. We will calculate the radius of convergence ρ of the generating function

$$Z_V(\xi) = \sum_{v \in \mathcal{V}} \xi^{|v|},$$

and deduce that $\mu \leq 1/\rho$.

The generating function of SAWs within a δ -kite is $\xi + 2\xi^2 + 2\xi^3$, so that, on taking into account the following a -step, this contributes a term $D(\xi) := \xi^2 + 2\xi^3 + 2\xi^4$ to Z_V . The other two non-trivial displacements within a given kite each contribute D also, and their combined generating function is $B(\xi) = 2D(\xi)$. By breaking a walk $v \in \mathcal{V}$ into sub-walks each using no more than 3 consecutive δ -kites followed by a kite which is not a δ -kite, we find that

$$Z_V(\xi) = \sum_{n=1}^{\infty} C(\xi)^n,$$

where $C(\xi) := (1 + D(\xi) + D(\xi)^2 + D(\xi)^3)B(\xi)$. Therefore, μ is bounded above by the reciprocal of the positive root of $C(\xi) = 1$.

5.2. Cases (b, c). We follow the proof of part (a) wherever possible. Let $z \in \{\beta, \sigma, \delta\}$. The orbital Schreier graph $\mathcal{S}_\omega(\neg z)$ of Γ_ω is obtained from $\mathcal{S}_\omega$ by deleting all elements labelled z ; see Figure 5.2 for an illustration of the cases $z = \beta, \delta$.

Let $\mathcal{W}_0(\neg z)$ be the set of finite, labelled walks on $\mathcal{S}_\omega(\neg z)$ starting at the root 1^∞ that, at each step, either move one step rightwards or pass around a loop. We view $\mathcal{W}_0$ as a set of finite words in the alphabet $\{a, \beta, \sigma, \delta\} \setminus \{z\}$ without consecutive repetitions.

As in part (a), each $w \in \mathcal{W}_0(\neg z)$ lifts to a distinct walk $\bar{w}$ on $G_\omega(\neg z)$, and furthermore Lemma 5.1 holds. We now examine the units of the Schreier graph $\mathcal{S}_\omega(\neg z)$, beginning with the case $z = \delta$.

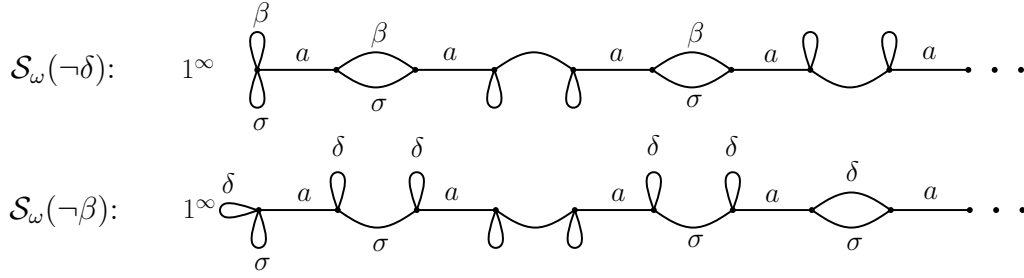


FIGURE 5.2. The one-ended orbital Schreier graphs $\mathcal{S}_w(-\delta)$ and $\mathcal{S}_w(-\beta)$ of the ray 1^∞ for the Grigorchuk group Γ_w with respective generator sets $\{a, \beta, \sigma\}$ and $\{a, \sigma, \delta\}$. The Schreier graph $\mathcal{S}_w(-\sigma)$ is similar to $\mathcal{S}_w(-\beta)$.

5.2.1. *The case $z = \delta$.* Let $\mathcal{W}$ be the subset of $\mathcal{W}_0(-\delta)$ containing words that end in a . As above, $\mathcal{W}$ lifts to a set $\overline{\mathcal{W}}$ of SAWs on G . It turns out that $\mathcal{W}$ is not sufficiently large to obtain an adequate lower bound for $\mu(G_w(-\delta))$, and therefore we shall augment $\mathcal{W}$ to a larger set $\mathcal{W}'$ of words. This will be done by considering the individual units of $\mathcal{S}_w(-\delta)$. The unit between r_i and r_{i+1} is called *odd* (respectively, *even*) if i is odd (respectively, even). It may be seen that every odd unit is a copy of the unit u_1 of Figure 5.3, and every even unit either is a copy of u_1 or contains a copy of u_2 (of the same figure).

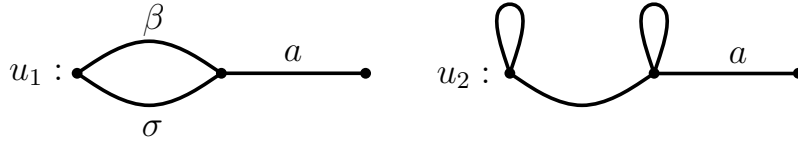


FIGURE 5.3. Two units contributing to $\mathcal{S}_w(-\delta)$, denoted u_1 (left) and u_2 (right). The omitted labels in u_2 depend on the position of the unit in the Schreier graph.

Let $\xi \geq 0$. As in (5.1)–(5.2), the generating function

$$(5.3) \quad Z(\xi) = \sum_{w \in \overline{\mathcal{W}}} \xi^{|w|}$$

of $\overline{\mathcal{W}}$ satisfies (for $\xi \geq 0$)

$$(5.4) \quad Z(\xi) \geq A_0(\xi) \sum_{n=0}^{\infty} \prod_{i=1}^n A_i(\xi),$$

where $A_0(\xi)$ is the generating function of walks from 1^∞ to r_1 , and

$$A_i(\xi) = \begin{cases} 2\xi^2 & \text{when } i \text{ is odd,} \\ \text{either } 2\xi^2 \text{ or } \xi^2 + 2\xi^3 + \xi^4 & \text{when } i \text{ is even.} \end{cases}$$

This can be improved by augmenting the set of walks in copies of the unit u_1 .

Walks from left to right in u_1 correspond to the set $L := \{\beta a, \sigma a\}$ of words. As in the corresponding part of the proof of part (a), we augment this set to the set $L' = \{\beta a, \sigma a, \beta\sigma\beta a, \sigma\beta\sigma a\}$. This has the effect of replacing $A_i(x)$, for odd i , by $\hat{A}(\xi) := 2\xi^2 + 2\xi^4$. Note that

$$(5.5) \quad \hat{A}(\xi) \geq \xi^2 + 2\xi^3 + \xi^4.$$

As in the proof of part (a), the walks in $G_\omega(\neg\delta)$ thus obtained are indeed SAWs.

Returning to (5.3)–(5.4), and grouping each odd unit with the following even unit, we deduce by (5.5) that

$$Z(\xi) \geq A_0(\xi) \sum_{n=0}^{\infty} H(\xi)^n$$

where

$$H(\xi) = \hat{A}(\xi)(\xi^2 + 2\xi^3 + \xi^4) = 2\xi^4(1 + \xi^2)(1 + \xi)^2.$$

Recalling (1.1), it follows that $\mu \geq \gamma \approx 1.635$ where $1/\gamma$ is the positive root of $H(\xi) = 1$. The lower bound of the theorem is proved.

Let $K_2 * \mathbb{Z}_4$ denote the free product (Fisher) graph of an edge and a 4-cycle (see the discussion around [?, Thm 7.2]). The upper bound $\mu \leq \mu(K_2 * \mathbb{Z}_4) \approx 1.899$ may be obtained by applying [?, Thm 7.2], on noting that $G_\omega(\neg\delta)$ has degree 3 and girth 4. That this upper bound is a *strict* inequality holds by the remark concerning the graph $K_2 * \mathbb{Z}_4$ in [?], and the strict-inequality theorem [?, Cor. 4.1]. This bound may be improved by using Mrazović's argument (see the proof of part (a)) with the given functions B and D .

5.2.2. Case (c). Since the arguments for $z = \beta, \sigma$ are essentially the same, we may suppose $z = \beta$. The orbital Schreier graph $\mathcal{S}_\omega(\neg\beta)$ is a line of copies of u_2 interspersed by a number of appearances of u_1 , with the last labelled as in Figure 5.4. The words represented in such a copy of u_1 are $\{\sigma a, \delta a\}$. We augment this set to $\{\sigma a, \delta a, \sigma\delta\sigma a, \delta\sigma\delta a\}$, as in the style of the last proof. As in (5.4), we deduce that

$$(5.6) \quad Z(\xi) \geq A_0(\xi) \sum_{n=0}^{\infty} \prod_{i=1}^n A_i(\xi),$$

where each A_i (with $i \geq 1$) is either $A_1(\xi) := 2\xi^2(1 + \xi^2)$ or $A_2(\xi) := \xi^2(1 + \xi)^2$. It is easily checked that $A_1(1/\phi), A_2(1/\phi) \geq 1$, so that $Z(1/\phi) = \infty$, whence, by (1.1), $\mu(G_\omega(\neg\beta)) \geq \phi$.

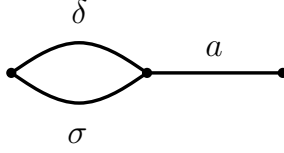


FIGURE 5.4. The labels on every copy of u_1 in $\mathcal{S}_\omega(\neg\beta)$. Note the missing loops of type β .

Turning to the upper bound on μ , we have that $\mu < \mu(K_2 * \mathbb{Z}_4)$ as in the proof of part (b). This bound may be improved using Mrazović's argument (from the proof of part (a)) to obtain $\mu \leq \gamma''$ with γ'' as given.

Corollary 4.2 provides a quantification of the number of appearances of u_1 , and hence an improvement over the lower bound derived above. Let π be the asymptotic proportion of units u_1 amongst units at vertices v_{4m} of $\mathcal{S}_\omega(\neg\beta)$ for $m \geq 1$. By Corollary 4.2,

$$(5.7) \quad \pi = \sum_{a=1}^{\infty} \frac{1}{2^a} 1(\lambda_{\omega_a} = \beta),$$

which is strictly positive whenever ω has the property that there exists $a \geq 1$ such that $\lambda_{\omega_a} = \beta$. When this holds, we may proceed as in (5.4) and (5.6) to find that $\mu(G_\omega(\neg\beta)) \geq \zeta$ where ζ is the reciprocal of the positive root of the equation

$$(5.8) \quad [A_1(\xi)]^\pi [A_1(\xi)]^{1-\pi} A_2(\xi) - 1 = 0.$$

This root is strictly smaller than $1/\phi$.

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DECLARATION

The author has no conflicts of interest, and no data arose in this work.

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