the appropriate boundary condition is $w(T, x) = (x - K)^+$. It is not always easy to find the solution, but there is a general method known as the 'Feynman–Kac formula', not discussed further here, which allows a representation of the solution in terms of a diffusion process. When the solution exists, the claim is said to be 'hedgeable', and the self-financing portfolio which replicates it is called the 'hedge'.

Proof. Assume that w satisfies (23), and set

$$\xi = w_x, \quad \psi = e^{-rt}(w - xw_x).$$

It is easily checked that the portfolio (ξ, ψ) has value function $w(t, S_t)$ and, via (23), that the pair ξ, ψ satisfy equations (16) and (17).

Conversely, if $w(t, S_t)$ is the value of a self-financing portfolio then $w(t, x) = \xi(t, x)x + \psi(t, x)e^{rt}$ for some pair ξ, ψ satisfying equations (16) and (17). We compute w_x and compare with (16) to find that $\xi = w_x$. Setting $\psi = e^{-rt}(w - xw_x)$, we substitute into (17) to deduce that equation (23) holds. ∎

Proof of Theorem (10). Finally we return to the proof of the Black–Scholes formula, showing first that the portfolio (α, β), given in equations (13), is self-financing. Set

$$\alpha(t, x) = \Phi(d_1(t, x)), \quad \beta(t, x) = -Ke^{-rT}\Phi(d_2(t, x)),$$

where d_1 and d_2 are given in (12). We note from (12) that

(24) $$d_2^2 = d_1^2 - 2\log(x/K) - 2r(T - t),$$

and it is straightforward to deduce by substitution that the pair α, β satisfy equations (16) and (17). Therefore, the portfolio (α, β) is self-financing. By construction, it has value function $V_t(\alpha, \beta)$ given in (11).

We may take the limit in (11) as $t \uparrow T$. Since

$$d_i(t, S_t) \to \begin{cases} -\infty & \text{if } S_T < K, \\ \infty & \text{if } S_T > K, \end{cases}$$

for $i = 1, 2$, we deduce that $V_T(\alpha, \beta) = (S_T - K)^+$ whenever $S_T \neq K$. Now $\mathbb{P}(S_T = K) = 0$, and therefore $V_T(\alpha, \beta) = (S_T - K)^+$ almost surely. It follows as required that the portfolio (α, β) replicates the European call option. ∎

Exercises for Section 13.10

In the absence of any contrary indication, W denotes a standard Wiener process, and $\mathcal{F}_t$ is the smallest σ-field containing all null events with respect to which every member of $\{W_u : 0 \le u \le t\}$ is measurable. The process $S_t = \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$ is a geometric Brownian motion, and $r \ge 0$ is the interest rate.

1. (a) Let Z have the $N(\gamma, \tau^2)$ distribution. Show that

$$\mathbb{E}\big((ae^Z - K)^+\big) = ae^{\gamma + \frac{1}{2}\tau^2}\Phi\left(\frac{\log(a/K) + \gamma}{\tau} + \tau\right) - K\Phi\left(\frac{\log(a/K) + \gamma}{\tau}\right)$$

where Φ is the $N(0, 1)$ distribution function.