**(2) Theorem. Itô's formula.** *If* $dX = \mu(t, X)\,dt + \sigma(t, X)\,dW$ *and* $Y_t = f(t, X_t)$, *where* $f$ *is twice continuously differentiable on* $[0, \infty) \times \mathbb{R}$, *then* $Y$ *is also an Itô process, given by*†

**(3)** $$dY = \big[f_x(t, X)\mu(t, X) + f_t(t, X) + \tfrac{1}{2} f_{xx}(t, X)\sigma^2(t, X)\big]\,dt + f_x(t, X)\sigma(t, X)\,dW.$$

This formula may be extended to cover multivariate diffusions. We do not prove Itô's formula at the level of generality of (2), instead specializing to the following special case when $X$ is the standard Wiener process. The differentiability assumption on the function $f$ may be weakened.

**(4) Theorem. Itô's simple formula.** *Let* $f(s, w)$ *be thrice continuously differentiable on* $[0, \infty) \times \mathbb{R}$, *and let* $W$ *be a standard Wiener process. The process* $Y_t = f(t, W_t)$ *is an Itô process with*

$$dY = \big[f_t(t, W) + \tfrac{1}{2} f_{ww}(t, W)\big]\,dt + f_w(t, W)\,dW.$$

**Sketch proof.** Let $n \ge 1$, $\delta = t/n$, and write $\Delta_j = W_{(j+1)\delta} - W_{j\delta}$. The idea is to express $f(t, W_t)$ as the sum

**(5)** $$\begin{aligned} f(t, W_t) - f(0, W_0) = &\sum_{j=0}^{n-1} \big[f\big((j+1)\delta, W_{(j+1)\delta}\big) - f(j\delta, W_{(j+1)\delta})\big] \\ &+ \sum_{j=0}^{n-1} \big[f(j\delta, W_{(j+1)\delta}) - f(j\delta, W_{j\delta})\big] \end{aligned}$$

and to use Taylor's theorem to study its behaviour as $n \to \infty$. We leave out the majority of details necessary to achieve this, presenting instead the briefest summary.

By (5) and Taylor's theorem, there exist random variables $\theta_j \in [j\delta, (j+1)\delta]$ and $\Omega_j \in [W_{j\delta}, W_{(j+1)\delta}]$ such that

**(6)** $$\begin{aligned} f(t, W_t) - f(0, W_0) = &\sum_{j=0}^{n-1} f_t(\theta_j, W_{(j+1)\delta})\delta + \sum_{j=0}^{n-1} f_w(j\delta, W_{j\delta})\Delta_j \\ &+ \tfrac{1}{2}\sum_{j=0}^{n-1} f_{ww}(j\delta, W_{j\delta})\Delta_j^2 + \tfrac{1}{6}\sum_{j=0}^{n-1} f_{www}(j\delta, \Omega_j)\Delta_j^3. \end{aligned}$$

We consider these terms one by one, as $n \to \infty$.

(i) It is a consequence of the continuity properties of $f$ and $W$ that

$$\sum_{j=0}^{n-1} f_t(\theta_j, W_{(j+1)\delta})\delta \xrightarrow{\text{a.s.}} \int_0^t f_t(s, W_s)\,ds.$$

(ii) Using the differentiability of $f$, one may see that $\sum_{j=0}^{n-1} f_w(j\delta, W_{j\delta})\Delta_j$ converges in mean square as $n \to \infty$ to the Itô integral $\int_0^t f_w(s, W_s)\,dW_s$.

†Here $f_t(t, X)$ and $f_x(t, X)$ denote the derivatives of $f$ with respect to its first and second arguments respectively, and evaluated at $(t, X_t)$.