(23) Theorem†. *Let Π be a Poisson process on $\mathbb{R}$ with constant intensity λ, let $r : \mathbb{R} \to \mathbb{R}$ be a smooth function, and let $\{W_x : x \in \Pi\}$ be independent identically distributed random variables, independent of Π. The sum*

$$G(t) = \sum_{x \in \Pi,\ x \geq 0} r(t - x) W_x$$

has characteristic function

$$\mathbb{E}(e^{i\theta G(t)}) = \exp\left\{ \lambda \int_0^t \left(\mathbb{E}(e^{i\theta W r(s)}) - 1 \right) ds \right\},$$

where W has the common distribution of the W_x. In particular,

$$\mathbb{E}(G(t)) = \lambda \mathbb{E}(W) \int_0^t r(s)\, ds.$$

Proof. This runs just like that of Theorem (6.12.11), which is in fact a special case. It is left as an *exercise* to check the details. The mean of $G(t)$ is calculated from its characteristic function by differentiating the latter at $\theta = 0$. ■

A similar idea works in higher dimensions, as the following demonstrates.

(24) Example. Olbers's paradox‡. Suppose that stars occur in $\mathbb{R}^3$ at the points $\{\mathbf{R}_i : i \geq 1\}$ of a Poisson process with constant intensity λ. The star at $\mathbf{R}_i$ has brightness B_i, where the B_i are independent and identically distributed with mean β. The intensity of the light striking an observer at the origin O from a star of brightness B, distance r away, is (in the absence of intervening clouds of dust) equal to cB/r^2, for some absolute constant c. Hence the total illumination at O from stars within a large ball S with radius a is

$$I_a = \sum_{i : |\mathbf{R}_i| \leq a} \frac{cB_i}{|\mathbf{R}_i|^2}.$$

Conditional on the event that the number N_a of such stars satisfies $N_a = n$, we have from the conditional property (11) that these n stars are uniformly and independently distributed over S. Hence

$$\mathbb{E}(I_a \mid N_a) = N_a c\beta \frac{1}{|S|} \int_S \frac{1}{|\mathbf{r}|^2}\, dV.$$

Now $\mathbb{E}(N_a) = \lambda |S|$, whence

$$\mathbb{E} I_a = \lambda c\beta \int_S \frac{1}{|\mathbf{r}|^2}\, dV = \lambda c\beta (4\pi a).$$

The fact that this is unbounded as $a \to \infty$ is called 'Olbers's paradox', and suggests that the celestial sphere should be uniformly bright at night. The fact that it is not is a problem whose resolution is still a matter for debate. One plausible explanation relies on a sufficiently fast rate of expansion of the Universe. ●

†This theorem is sometimes called the Campbell–Hardy theorem. See also Exercise (6.13.2).

‡Put to Newton and Halley by William Stukeley in 1721, more than a century before Olbers's work.