6.7 Branching processes revisited

The foregoing general theory is an attractive and concise account of the evolution through time of a Markov chain. Unfortunately, it is an inadequate description of many specific Markov chains. Consider for example a branching process $\{Z_0, Z_1, \dots\}$ where $Z_0 = 1$. If there is strictly positive probability $\mathbb{P}(Z_1 = 0)$ that each family is empty then 0 is an absorbing state. Hence 0 is persistent non-null, and all other states are transient. The chain is not irreducible but there exists a unique stationary distribution $\boldsymbol{\pi}$ given by $\pi_0 = 1$, $\pi_i = 0$ if $i > 0$. These facts tell us next to nothing about the behaviour of the process, and we must look elsewhere for detailed information. The difficulty is that the process may behave in one of various qualitatively different ways depending, for instance, on whether or not it ultimately becomes extinct. One way of approaching the problem is to study the behaviour of the process *conditional* upon the occurrence of some event, such as extinction, or on the value of some random variable, such as the total number $\sum_i Z_i$ of progeny. This section contains an outline of such a method.

Let f and G be the mass function and generating function of a typical family size Z_1:

$$f(k) = \mathbb{P}(Z_1 = k), \qquad G(s) = \mathbb{E}(s^{Z_1}).$$

Let $T = \inf\{n : Z_n = 0\}$ be the time of extinction, with the convention that the infimum of the empty set is $+\infty$. Roughly speaking, if $T = \infty$ then the process will grow beyond all possible bounds, whilst if $T < \infty$ then the size of the process never becomes very large and subsequently reduces to zero. Think of $\{Z_n\}$ as a fluctuating sequence which either becomes so large that it escapes to ∞ or is absorbed at 0 during one of its fluctuations. From the results of Section 5.4, the probability $\mathbb{P}(T < \infty)$ of ultimate extinction is the smallest non-negative root of the equation $s = G(s)$. Now let

$$E_n = \{n < T < \infty\}$$

be the event that extinction occurs at some time after n. We shall study the distribution of Z_n conditional upon the occurrence of E_n. Let

$${}_0p_j(n) = \mathbb{P}(Z_n = j \mid E_n)$$

be the conditional probability that $Z_n = j$ given the future extinction of Z. We are interested in the limiting value

$${}_0\pi_j = \lim_{n\to\infty} {}_0p_j(n),$$

if this limit exists. To avoid certain trivial cases we assume henceforth that

$$0 < f(0) + f(1) < 1, \qquad f(0) > 0;$$

these conditions imply for example that $0 < \mathbb{P}(E_n) < 1$ and that the probability η of ultimate extinction satisfies $0 < \eta \le 1$.