## Exercises for Section 1.4

**1.** Prove that $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\mathbb{P}(A)/\mathbb{P}(B)$ whenever $\mathbb{P}(A)\mathbb{P}(B) \neq 0$. Show that, if $\mathbb{P}(A \mid B) > \mathbb{P}(A)$, then $\mathbb{P}(B \mid A) > \mathbb{P}(B)$.

**2.** For events $A_1, A_2, \dots, A_n$ satisfying $\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_{n-1}) > 0$, prove that

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

**3.** A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head? He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head? He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

**4.** What do you think of the following 'proof' by Lewis Carroll that an urn cannot contain two balls of the same colour? Suppose that the urn contains two balls, each of which is either black or white; thus, in the obvious notation, $\mathbb{P}(\mathrm{BB}) = \mathbb{P}(\mathrm{BW}) = \mathbb{P}(\mathrm{WB}) = \mathbb{P}(\mathrm{WW}) = \frac{1}{4}$. We add a black ball, so that $\mathbb{P}(\mathrm{BBB}) = \mathbb{P}(\mathrm{BBW}) = \mathbb{P}(\mathrm{BWB}) = \mathbb{P}(\mathrm{BWW}) = \frac{1}{4}$. Next we pick a ball at random; the chance that the ball is black is (using conditional probabilities) $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{2}{3}$. However, if there is probability $\frac{2}{3}$ that a ball, chosen randomly from three, is black, then there must be two black and one white, which is to say that originally there was one black and one white ball in the urn.

**5. The Monty Hall problem: goats and cars.** (a) In a game show; you have to choose one of three doors. One conceals a new car, two conceal old goats. You choose, but your chosen door is not opened immediately. Instead the presenter opens another door, which reveals a goat. He offers you the opportunity to change your choice to the third door (unopened and so far unchosen). Let $p$ be the (conditional) probability that the third door conceals the car. The presenter's protocol is:

(i) he is determined to show you a goat; with a choice of two, he picks one at random. Show $p = \frac{2}{3}$.
(ii) he is determined to show you a goat; with a choice of two goats (Bill and Nan, say) he shows you Bill with probability $b$. Show that, given you see Bill, the probability is $1/(1+b)$.
(iii) he opens a door chosen at random irrespective of what lies behind. Show $p = \frac{1}{2}$.

(b) Show that, for $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, there exists a protocol such that $p = \alpha$. Are you well advised to change your choice to the third door?

(c) In a variant of this question, the presenter is permitted to open the first door chosen, and to reward you with whatever lies behind. If he chooses to open another door, then this door invariably conceals a goat. Let $p$ be the probability that the unopened door conceals the car, conditional on the presenter having chosen to open a second door. Devise protocols to yield the values $p = 0$, $p = 1$, and deduce that, for any $\alpha \in [0, 1]$, there exists a protocol with $p = \alpha$.

**6. The prosecutor's fallacy†.** Let $G$ be the event that an accused is guilty, and $T$ the event that some testimony is true. Some lawyers have argued on the assumption that $\mathbb{P}(G \mid T) = \mathbb{P}(T \mid G)$. Show that this holds if and only if $\mathbb{P}(G) = \mathbb{P}(T)$.

**7. Urns.** There are $n$ urns of which the $r$th contains $r-1$ red balls and $n-r$ magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

(a) the second ball is magenta;
(b) the second ball is magenta, given that the first is magenta.

†The prosecution made this error in the famous Dreyfus case of 1894.