6. Let X and Y have the standard bivariate normal density function of Exercise (5), and define $Z = \max\{X, Y\}$. Show that $\mathbb{E}(Z) = \sqrt{(1-\rho)/\pi}$, and $\mathbb{E}(Z^2) = 1$.

7. Let X and Y be independent exponential random variables with parameters λ and μ. Show that $Z = \min\{X, Y\}$ is independent of the event $\{X < Y\}$. Find:
(a) $\mathbb{P}(X = Z)$,
(b) the distributions of $U = \max\{X - Y, 0\}$, denoted $(X - Y)^+$, and $V = \max\{X, Y\} - \min\{X, Y\}$,
(c) $\mathbb{P}(X \le t < X + Y)$ where $t > 0$.

8. A point (X, Y) is picked at random uniformly in the unit circle. Find the joint density of R and X, where $R^2 = X^2 + Y^2$.

9. A point (X, Y, Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint density of Z and R, where $R^2 = X^2 + Y^2 + Z^2$.

10. Let X and Y be independent and exponentially distributed with parameters λ and μ. Find the joint distribution of $S = X + Y$ and $R = X/(X + Y)$. What is the density of R?

11. Find the density of $Y = a/(1 + X^2)$, where X has the Cauchy distribution.

12. Let (X, Y) have the bivariate normal density of Exercise (5) with $0 \le \rho < 1$. Show that

$$[1 - \Phi(a)][1 - \Phi(c)] \le \mathbb{P}(X > a,\ Y > b) \le [1 - \Phi(a)][1 - \Phi(c)] + \frac{\rho\phi(b)[1 - \Phi(d)]}{\phi(a)},$$

where $c = (b - \rho a)/\sqrt{1-\rho^2}$, $d = (a - \rho b)/\sqrt{1-\rho^2}$, and ϕ and Φ are the density and distribution function of the $N(0, 1)$ distribution.

13. Let X have the Cauchy distribution. Show that $Y = X^{-1}$ has the Cauchy distribution also. Find another non-trivial distribution with this property of invariance.

14. Let X and Y be independent and gamma distributed as $\Gamma(\lambda, \alpha)$, $\Gamma(\lambda, \beta)$ respectively. Show that $W = X+Y$ and $Z = X/(X+Y)$ are independent, and that Z has the beta distribution with parameters α, β.

4.8 Sums of random variables

This section contains an important result which is a very simple application of the change of variable technique.

(1) Theorem. *If X and Y have joint density function f then $X + Y$ has density function*

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx.$$

Proof. Let $A = \{(x, y) : x + y \le z\}$. Then

$$\begin{aligned}\mathbb{P}(X + Y \le z) &= \iint_A f(u, v)\, du\, dv = \int_{u=-\infty}^{\infty} \int_{v=-\infty}^{z-u} f(u, v)\, dv\, du \\ &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z} f(x, y - x)\, dy\, dx\end{aligned}$$

by the substitution $x = u$, $y = v + u$. Reverse the order of integration to obtain the result. ∎