by a consideration of the minimum value of W on $(0, t]$. It follows that the density function of $W(t)$, conditional on $Z \cap \{W(0) = w\}$, where $w > 0$, is

$$h_w(x) = \frac{f(t, x - w) - f(t, x + w)}{\int_{-w}^{w} f(t, y)\, dy}, \qquad x > 0.$$

Divide top and bottom by $2w$, and take the limit as $w \downarrow 0$:

$$\lim_{w \downarrow 0} h_w(x) = -\frac{1}{f(t, 0)} \frac{\partial f}{\partial x} = \frac{x}{t} e^{-x^2/(2t)}, \qquad x > 0.$$

2. It is a standard exercise that, for a Wiener process W,

$$\mathbb{E}\{W(t) \mid W(s) = a, W(1) = 0\} = a\left(\frac{1-t}{1-s}\right),$$
$$\mathbb{E}\{W(s)^2 \mid W(0) = W(1) = 0\} = s(1-s),$$

if $0 \le s \le t \le 1$. Therefore the Brownian bridge B satisfies, for $0 \le s \le t \le 1$,

$$\mathbb{E}(B(s)B(t)) = \mathbb{E}\{B(s)\mathbb{E}(B(t) \mid B(s))\} = \frac{1-t}{1-s}\mathbb{E}(B(s)^2) = s(1-t)$$

as required. Certainly $\mathbb{E}(B(s)) = 0$ for all s, by symmetry.

3. $\widehat{W}$ is a zero-mean Gaussian process on $[0, 1]$ with continuous sample paths, and also $\widehat{W}(0) = \widehat{W}(1) = 0$. Therefore $\widehat{W}$ is a Brownian bridge if it has the same autocovariance function as the Brownian bridge, that is, $c(s, t) = \min\{s, t\} - st$. For $s < t$,

$$\operatorname{cov}(\widehat{W}(s), \widehat{W}(t)) = \operatorname{cov}(W(s) - sW(1), W(t) - tW(1)) = s - ts - st + st = s - st$$

since $\operatorname{cov}(W(u), W(v)) = \min\{u, v\}$. The claim follows.

4. Either calculate the instantaneous mean and variance of $\widetilde{W}$, or repeat the argument in the solution to Exercise (13.6.3). The only complication in this case is the necessity to show that $\widetilde{W}(t)$ is a.s. continuous at $t = 1$, i.e., that $u^{-1}W(u-1) \to 0$ a.s. as $u \to \infty$. There are various ways to show this. Certainly it is true in the limit as $u \to \infty$ through the integers, since, for integral u, $W(u-1)$ may be expressed as the sum of $u-1$ independent $N(0, 1)$ variables (use the strong law). It remains to fill in the gaps. Let n be a positive integer, let $x > 0$, and write $M_n = \max\{|W(u) - W(n)| : n \le u \le n+1\}$. We have by the stationarity of the increments that

$$\sum_{n=0}^{\infty} \mathbb{P}(M_n \ge nx) = \sum_{n=0}^{\infty} \mathbb{P}(M_1 \ge nx) \le 1 + \frac{\mathbb{E}(M_1)}{x} < \infty,$$

implying by the Borel–Cantelli lemma that $n^{-1}M_n \le x$ for all but finitely many values of n, a.s. Therefore $n^{-1}M_n \to 0$ a.s. as $n \to \infty$, implying that

$$\lim_{u \to \infty} \frac{1}{u+1}|W(u)| \le \lim_{n \to \infty} \frac{1}{n}\{|W(n)| + M_n\} \to 0 \qquad \text{a.s.}$$

5. In the notation of Exercise (13.6.4), we are asked to calculate the probability that W has no zeros in the time interval between $s/(1-s)$ and $t/(1-t)$. By Theorem (13.4.8), this equals

$$1 - \frac{2}{\pi}\cos^{-1}\sqrt{\frac{s(1-t)}{t(1-s)}} = \frac{2}{\pi}\cos^{-1}\sqrt{\frac{t-s}{t(1-s)}}.$$