It may be checked that X' and Y' have the same distributions as X and Y, and furthermore, $\mathbb{P}(X' \neq Y') = \mathbb{P}(Z = 0) = 1 - \mu = \frac{1}{2} d_{\mathrm{TV}}(X, Y)$.

(e) By part (d), we may find independent pairs (X_i', Y_i'), $1 \le i \le n$, having the same marginals as (X_i, Y_i), respectively, and such that $\mathbb{P}(X_i' \neq Y_i') = \frac{1}{2} d_{\mathrm{TV}}(X_i, Y_i)$. Now,

$$\begin{aligned} d_{\mathrm{TV}}\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right) &= d_{\mathrm{TV}}\left(\sum_{i=1}^n X_i', \sum_{i=1}^n Y_i'\right) \\ &\le 2\mathbb{P}\left(\sum_{i=1}^n X_i' \neq \sum_{i=1}^n Y_i'\right) \le 2\sum_{i=1}^n \mathbb{P}(X_i' \neq Y_i') = \sum_{i=1}^n d_{\mathrm{TV}}(X_i, Y_i). \end{aligned}$$

17. If $X_1, X_2, \dots$ are independent variables having the Poisson distribution with parameter λ, then $S_n = X_1 + X_2 + \cdots + X_n$ has the Poisson distribution with parameter λn. Now $n^{-1} S_n \xrightarrow{\mathrm{D}} \lambda$, so that $\mathbb{E}(g(n^{-1} S_n)) \to g(\lambda)$ for all bounded continuous g. The result follows.

18. The characteristic function ψ_{mn} of

$$U_{mn} = \frac{(X_n - n) - (Y_m - m)}{\sqrt{m+n}}$$

satisfies

$$\log \psi_{mn}(t) = n\left(e^{it/\sqrt{m+n}} - 1\right) + m\left(e^{-it/\sqrt{m+n}} - 1\right) + \frac{(m-n)it}{\sqrt{m+n}} \to -\tfrac{1}{2}t^2$$

as $m, n \to \infty$, implying by the continuity theorem that $U_{mn} \xrightarrow{\mathrm{D}} N(0, 1)$. Now $X_n + Y_m$ is Poisson-distributed with parameter $m + n$, and therefore

$$V_{mn} = \sqrt{\frac{X_n + Y_m}{m+n}} \xrightarrow{\mathrm{P}} 1 \qquad \text{as } m, n \to \infty$$

by the law of large numbers and Problem (3). It follows by Slutsky's theorem (7.2.5a) that $U_{mn} / V_{mn} \xrightarrow{\mathrm{D}} N(0, 1)$ as required.

19. (a) The characteristic function of X_n is $\phi_n(t) = \exp\{i\mu_n t - \frac{1}{2}\sigma_n^2 t^2\}$ where μ_n and σ_n^2 are the mean and variance of X_n. Now, $\lim_{n\to\infty} \phi_n(1)$ exists. However $\phi_n(1)$ has modulus $e^{-\frac{1}{2}\sigma_n^2}$, and therefore $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ exists. The remaining component $e^{i\mu_n t}$ of $\phi_n(t)$ converges as $n \to \infty$, say $e^{i\mu_n t} \to \theta(t)$ as $n \to \infty$ where $\theta(t)$ lies on the unit circle of the complex plane. Now $\phi_n(t) \to \theta(t) e^{-\frac{1}{2}\sigma^2 t^2}$, which is required to be a characteristic function; therefore θ is a continuous function of t. Of the various ways of showing that $\theta(t) = e^{i\mu t}$ for some μ, here is one. The sequence $\psi_n(t) = e^{i\mu_n t}$ is a sequence of characteristic functions whose limit $\theta(t)$ is continuous at $t = 0$. Therefore θ is a characteristic function. However ψ_n is the characteristic function of the constant μ_n, which must converge in distribution as $n \to \infty$; it follows that the real sequence $\{\mu_n\}$ converges to some limit μ, and $\theta(t) = e^{i\mu t}$ as required.

This proves that $\phi_n(t) \to \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}$, and therefore the limit X is $N(\mu, \sigma^2)$.

(b) Each linear combination $sX_n + tY_n$ converges in probability, and hence in distribution, to $sX + tY$. Now $sX_n + tY_n$ has a normal distribution, implying by part (a) that $sX + tY$ is normal. Therefore the joint characteristic function of X and Y satisfies

$$\begin{aligned} \phi_{X,Y}(s, t) = \phi_{sX+tY}(1) &= \exp\left\{i\mathbb{E}(sX + tY) - \tfrac{1}{2}\operatorname{var}(sX + tY)\right\} \\ &= \exp\left\{i(s\mu_X + t\mu_Y) - \tfrac{1}{2}(s^2\sigma_X^2 + 2st\rho_{XY}\sigma_X\sigma_Y + t^2\sigma_Y^2)\right\} \end{aligned}$$