where $\mathbb{E}(s^{W_r}) = G_r(s)$ is given in Problem (5.12.46).

48. We have that

$$X_{n+1} = \begin{cases} \frac{1}{2}X_n & \text{with probability } \frac{1}{2}, \\ \frac{1}{2}X_n + Y_n & \text{with probability } \frac{1}{2}. \end{cases}$$

Hence the characteristic functions satisfy

$$\begin{aligned} \phi_{n+1}(t) = \mathbb{E}(e^{itX_{n+1}}) &= \tfrac{1}{2}\phi_n(\tfrac{1}{2}t) + \tfrac{1}{2}\phi_n(\tfrac{1}{2}t)\frac{\lambda}{\lambda - it} \\ &= \phi_n(\tfrac{1}{2}t)\frac{\lambda - \frac{1}{2}it}{\lambda - it} = \phi_{n-1}(\tfrac{1}{4}t)\frac{\lambda - \frac{1}{4}it}{\lambda - it} = \cdots = \phi_1(t2^{-n})\frac{\lambda - it2^{-n}}{\lambda - it} \to \frac{\lambda}{\lambda - it} \end{aligned}$$

as $n \to \infty$. The limiting distribution is exponential with parameter λ.

49. We have that

$$\int_0^1 G(s)\,ds = \mathbb{E}\left(\int_0^1 s^X\,ds\right) = \mathbb{E}\left(\left.\frac{s^{X+1}}{X+1}\right|_0^1\right) = \mathbb{E}\left(\frac{1}{X+1}\right).$$

(a) $(1-e^{-\lambda})/\lambda$, (b) $-(p/q^2)(q+\log p)$, (c) $(1-q^{n+1})/[(n+1)p]$, (d) $-[1+(p/q)\log p]/\log p$.
(e) Not if $\mathbb{P}(X+1>0)=1$, by Jensen's inequality (see Exercise (5.6.1)) and the strict concavity of the function $f(x) = 1/x$. If $Y = X+1$ is permitted to be negative, consider the case when $\mathbb{P}(Y=1) = p$, $\mathbb{P}(Y=-2) = \mathbb{P}(Y=-\frac{1}{2}) = \frac{1}{2}(1-p)$ for suitable p.

50. By compounding, as in Theorem (5.1.25), the sum has characteristic function

$$G_N(\phi_X(t)) = \frac{p\phi_X(t)}{1 - q\phi_X(t)} = \frac{\lambda p}{\lambda p - it},$$

whence the sum is exponentially distributed with parameter λp.

51. Consider the function $G(x) = \{\mathbb{E}(X^2)\}^{-1}\int_{-\infty}^x y^2\,dF(y)$. This function is right-continuous and increases from 0 to 1, and is therefore a distribution function. Its characteristic function is

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{\mathbb{E}(X^2)} x^2\,dF(x) = -\frac{1}{\mathbb{E}(X^2)}\frac{d^2}{dt^2}\phi(t).$$

52. By integration, $f_X(x) = f_Y(y) = \frac{1}{2}$, $|x|<1$, $|y|<1$. Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent. Now,

$$f_{X+Y}(z) = \int_{-1}^1 f(x, z-x)\,dx = \begin{cases} \frac{1}{4}(z+2) & \text{if } -2<z<0, \\ \frac{1}{4}(2-z) & \text{if } 0<z<2, \end{cases}$$

the 'triangular' density function on $(-2,2)$. This is the density function of the sum of two independent random variables uniform on $(-1,1)$.