if $\mathbb{P}(A)\mathbb{P}(B) \neq 0$. Hence

$$\frac{\mathbb{P}(A \mid B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)},$$

whence the last part is immediate.

2. Set $A_0 = \Omega$ for notational convenience. Expand each term on the right-hand side to obtain

$$\prod_{r=1}^{n} \mathbb{P}\Big(A_r \,\Big|\, \bigcap_{k=1}^{r-1} A_k\Big) = \prod_{r=1}^{n} \frac{\mathbb{P}(\bigcap_1^r A_k)}{\mathbb{P}(\bigcap_1^{r-1} A_k)} = \mathbb{P}\Big(\bigcap_1^n A_k\Big).$$

3. Let M be the event that the first coin is double-headed, R the event that it is double-tailed, and N the event that it is normal. Let H_{l}^i be the event that the lower face is a head on the ith toss, T_{u}^i the event that the upper face is a tail on the ith toss, and so on. Then, using conditional probability *ad nauseam*, we find:

(i) $$\mathbb{P}(H_{\mathrm{l}}^1) = \tfrac{2}{5}\mathbb{P}(H_{\mathrm{l}}^1 \mid M) + \tfrac{1}{5}\mathbb{P}(H_{\mathrm{l}}^1 \mid R) + \tfrac{2}{5}\mathbb{P}(H_{\mathrm{l}}^1 \mid N) = \tfrac{2}{5} + 0 + \tfrac{2}{5}\cdot\tfrac{1}{2} = \tfrac{3}{5}.$$

(ii) $$\mathbb{P}(H_{\mathrm{l}}^1 \mid H_{\mathrm{u}}^1) = \frac{\mathbb{P}(H_{\mathrm{l}}^1 \cap H_{\mathrm{u}}^1)}{\mathbb{P}(H_{\mathrm{u}}^1)} = \frac{\mathbb{P}(M)}{\mathbb{P}(H_{\mathrm{l}}^1)} = \tfrac{2}{5}\big/\tfrac{3}{5} = \tfrac{2}{3}.$$

(iii) $$\begin{aligned}\mathbb{P}(H_{\mathrm{l}}^2 \mid H_{\mathrm{u}}^1) &= 1\cdot\mathbb{P}(M \mid H_{\mathrm{u}}^1) + \tfrac{1}{2}\mathbb{P}(N \mid H_{\mathrm{u}}^1)\\ &= \mathbb{P}(H_{\mathrm{l}}^1 \mid H_{\mathrm{u}}^1) + \tfrac{1}{2}\big(1 - \mathbb{P}(H_{\mathrm{l}}^1 \mid H_{\mathrm{u}}^1)\big) = \tfrac{2}{3} + \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{5}{6}.\end{aligned}$$

(iv)
$$\mathbb{P}(H_{\mathrm{l}}^2 \mid H_{\mathrm{u}}^1 \cap H_{\mathrm{u}}^2) = \frac{\mathbb{P}(H_{\mathrm{l}}^2 \cap H_{\mathrm{u}}^1 \cap H_{\mathrm{u}}^2)}{\mathbb{P}(H_{\mathrm{u}}^1 \cap H_{\mathrm{u}}^2)} = \frac{\mathbb{P}(M)}{1\cdot\mathbb{P}(M) + \frac{1}{4}\cdot\mathbb{P}(N)} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{10}} = \frac{4}{5}.$$

(v) From (iv), the probability that he discards a double-headed coin is $\frac{4}{5}$, the probability that he discards a normal coin is $\frac{1}{5}$. (There is of course no chance of it being double-tailed.) Hence, by conditioning on the discard,

$$\mathbb{P}(H_{\mathrm{u}}^3) = \tfrac{4}{5}\mathbb{P}(H_{\mathrm{u}}^3 \mid M) + \tfrac{1}{5}\mathbb{P}(H_{\mathrm{u}}^3 \mid N) = \tfrac{4}{5}\big(\tfrac{1}{4} + \tfrac{1}{2}\cdot\tfrac{1}{2}\big) + \tfrac{1}{5}\big(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{4}\big) = \tfrac{21}{40}.$$

4. The final calculation of $\frac{2}{3}$ refers not to a *single* draw of one ball from an urn containing three, but rather to a composite experiment comprising more than one stage (in this case, two stages). While it is true that {two black, one white} is the only fixed collection of balls for which a random choice is black with probability $\frac{2}{3}$, the composition of the urn is *not determined* prior to the final draw.

After all, if Carroll's argument were correct then it would apply also in the situation when the urn originally contains just one ball, either black or white. The final probability is now $\frac{3}{4}$, implying that the original ball was one half black and one half white! Carroll was himself aware of the fallacy in this argument.

5. (a) One cannot compute probabilities without knowing the rules governing the conditional probabilities. We assume that each of the 6 orderings of the car and goats are equally likely. Let C_i be the event that the ith door conceals the car, G the event that you see a goat, and B the event that you see Bill.

(i) We have that

$$\mathbb{P}(C_3 \mid G) = \frac{\mathbb{P}(C_3 \cap G \mid C_1)\mathbb{P}(C_1) + \mathbb{P}(C_3 \cap G \mid C_1^{\mathrm{c}})\mathbb{P}(C_1^{\mathrm{c}})}{\mathbb{P}(G \mid C_1)\mathbb{P}(C_1) + \mathbb{P}(G \mid C_1^{\mathrm{c}})\mathbb{P}(C_1^{\mathrm{c}})} = \frac{0\cdot\frac{1}{3} + 1\cdot\frac{2}{3}}{1\cdot\frac{1}{3} + 1\cdot\frac{2}{3}} = \frac{2}{3}.$$

(ii) Use a similar formula, and note that $\mathbb{P}(C_3 \cap B \mid C_1^c) = \mathbb{P}(B \mid C_1^c) = \frac{1}{2}$.

(iii) This time, $\mathbb{P}(C_3 \cap G \mid C_1^c) = \mathbb{P}(G \mid C_1^c) = \frac{1}{2}$.

(b) Let $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, and suppose the presenter possesses a coin which falls with heads upwards with probability $\beta = 6\alpha - 3$. He flips the coin before the show, and adopts strategy (i) if and only if the coin shows heads, and otherwise strategy (iii). The probability in question is now $\frac{2}{3}\beta + \frac{1}{2}(1-\beta) = \alpha$. You never lose by swapping, but whether you gain depends on the presenter's protocol.

(c) Let D denote the first door chosen, and consider the following protocols:

(iv) If D conceals a goat, open it. Otherwise open one of the other two doors at random. In this case $p = 0$.

(v) If D conceals the car, open it. Otherwise open the unique remaining door which conceals a goat. In this case $p = 1$.

As in part (b), a randomized algorithm provides the protocol necessary for the last part.

6. This is immediate by the definition of conditional probability.

7. Let C_i be the colour of the ith ball picked, and use the obvious notation.

(a) Since each urn contains the same number $n-1$ of balls, the second ball picked is equally likely to be any of the $n(n-1)$ available. One half of these balls are magenta, whence $\mathbb{P}(C_2 = \mathrm{M}) = \frac{1}{2}$.

(b) By conditioning on the choice of urn,

$$\mathbb{P}(C_2 = \mathrm{M} \mid C_1 = \mathrm{M}) = \frac{\mathbb{P}(C_1, C_2 = \mathrm{M})}{\mathbb{P}(C_1 = \mathrm{M})} = \sum_{r=1}^{n} \frac{(n-r)(n-r-1)}{n(n-1)(n-2)} \Big/ \frac{1}{2} = \frac{2}{3}.$$

1.5 Solutions. Independence

1. Clearly

$$\begin{aligned}\mathbb{P}(A^c \cap B) &= \mathbb{P}(B \setminus \{A \cap B\}) = \mathbb{P}(B) - \mathbb{P}(A \cap B)\\ &= \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A^c)\mathbb{P}(B).\end{aligned}$$

For the final part, apply the first part to the pair B, A^c.

2. Suppose $i < j$ and $m < n$. If $j < m$, then A_{ij} and A_{mn} are determined by distinct independent rolls, and are therefore independent. For the case $j = m$ we have that

$$\begin{aligned}\mathbb{P}(A_{ij} \cap A_{jn}) &= \mathbb{P}(i\text{th}, j\text{th, and } n\text{th rolls show same number})\\ &= \sum_{r=1}^{6} \tfrac{1}{6}\mathbb{P}\big(j\text{th and } n\text{th rolls both show } r \,\big|\, i\text{th shows } r\big) = \tfrac{1}{36} = \mathbb{P}(A_{ij})\mathbb{P}(A_{jn}),\end{aligned}$$

as required. However, if $i \neq j \neq k$, $\mathbb{P}(A_{ij} \cap A_{jk} \cap A_{ik}) = \frac{1}{36} \neq \frac{1}{216} = \mathbb{P}(A_{ij})\mathbb{P}(A_{jk})\mathbb{P}(A_{ik})$.

3. That (a) implies (b) is trivial. Suppose then that (b) holds. Consider the outcomes numbered $i_1, i_2, \ldots, i_m$, and let $u_j \in \{\mathrm{H}, \mathrm{T}\}$ for $1 \le j \le m$. Let S_j be the set of all sequences of length $M = \max\{i_j : 1 \le j \le m\}$ showing u_j in the i_jth position. Clearly $|S_j| = 2^{M-1}$ and $\left|\bigcap_j S_j\right| = 2^{M-m}$. Therefore,

$$\mathbb{P}(S_j) = \frac{2^{M-1}}{2^M} = \frac{1}{2}, \qquad \mathbb{P}\Big(\bigcap_j S_j\Big) = \frac{2^{M-m}}{2^M} = \frac{1}{2^m},$$