13.9 Exercises. Itô's formula

In the absence of any contrary indication, W denotes a standard Wiener process, and $\mathcal{F}_t$ is the smallest σ-field containing all null events with respect to which every member of $\{W_u : 0 \le u \le t\}$ is measurable.

1. Let X and Y be independent standard Wiener processes. Show that, with $R_t^2 = X_t^2 + Y_t^2$,

$$Z_t = \int_0^t \frac{X_s}{R_s}\,dX_s + \int_0^t \frac{Y_s}{R_s}\,dY_s$$

is a Wiener process. [Hint: Use Theorem (13.8.13).] Hence show that R^2 satisfies

$$R_t^2 = 2\int_0^t R_s\,dW_s + 2t.$$

Generalize this conclusion to n dimensions.

2. Write down the SDE obtained via Itô's formula for the process $Y_t = W_t^4$, and deduce that $\mathbb{E}(W_t^4) = 3t^2$.

3. Show that $Y_t = tW_t$ is an Itô process, and write down the corresponding SDE.

4. Wiener process on a circle. Let $Y_t = e^{iW_t}$. Show that $Y = X_1 + iX_2$ is a process on the unit circle satisfying

$$dX_1 = -\tfrac{1}{2}X_1\,dt - X_2\,dW, \quad dX_2 = -\tfrac{1}{2}X_2\,dt + X_1\,dW.$$

5. Find the SDEs satisfied by the processes:
(a) $X_t = W_t/(1+t)$,
(b) $X_t = \sin W_t$,
(c) [Wiener process on an ellipse] $X_t = a\cos W_t$, $Y_t = b\sin W_t$, where $ab \ne 0$.

13.10 Exercises. Option pricing

In the absence of any contrary indication, W denotes a standard Wiener process, and $\mathcal{F}_t$ is the smallest σ-field containing all null events with respect to which every member of $\{W_u : 0 \le u \le t\}$ is measurable. The process $S_t = \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$ is a geometric Brownian motion, and $r \ge 0$ is the interest rate.

1. (a) Let Z have the $N(\gamma, \tau^2)$ distribution. Show that

$$\mathbb{E}\big((ae^Z - K)^+\big) = ae^{\gamma+\frac{1}{2}\tau^2}\,\Phi\left(\frac{\log(a/K)+\gamma}{\tau} + \tau\right) - K\,\Phi\left(\frac{\log(a/K)+\gamma}{\tau}\right)$$

where Φ is the $N(0,1)$ distribution function.

(b) Let $\mathbb{Q}$ be a probability measure under which σW is a Wiener process with drift $r - \mu$ and instantaneous variance σ^2. Show for $0 \le t \le T$ that

$$\mathbb{E}_{\mathbb{Q}}\big((S_T - K)^+ \,\big|\, \mathcal{F}_t\big) = S_t e^{r(T-t)}\Phi(d_1(t, S_t)) - K\,\Phi(d_2(t, S_t))$$

where

$$d_1(t,x) = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t,x) = d_1(t,x) - \sigma\sqrt{T-t}.$$