3. PERIODIC PERCOLATION PROBLEMS.

3.1. Introduction of probability. Site vs bond problems.

Let $\mathcal{G}$ be a graph satisfying (2.1)-(2.5) with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. The most classical percolation model is the one in which all bonds of $\mathcal{G}$ are randomly assigned to one of two classes, all bonds being assigned independently of each other. This is called bond-percolation, and the two kinds of bonds are called the passable or open bonds and the blocked or closed bonds. Instead of partitioning the bonds one often partitions the sites into two classes. Again all sites are assigned to one class or the other independently of each other. One now speaks of site-percolation and uses occupied and vacant sites to denote the two kinds of sites. The crucial requirement in both models is the independence of the bonds or sites, respectively. This makes the states of the bonds or sites into a family of independent two-valued random variables. Accordingly the above models are called Bernoulli-percolation models.

Formally one describes the models as follows. One denotes the possible configurations of the bonds (sites) by $+1$ and -1 with $+1$ standing for passable (occupied) and -1 for blocked (vacant). The configuration space for the whole system is then

(3.1) $$\Omega_{\mathcal{E}} = \prod_{\mathcal{E}}\{-1,+1\} \quad \text{or} \quad \Omega_{\mathcal{V}} = \prod_{\mathcal{V}}\{-1,+1\}$$

A generic point of $\Omega_{\mathcal{E}}$ is denoted by $\omega = \{\omega(e)\}_{e \in \mathcal{E}}$ and for the σ-field $\mathcal{B}_{\mathcal{E}}$ in $\Omega_{\mathcal{E}}$ we take σ-field generated by the cylinder sets of $\Omega_{\mathcal{E}}$, i.e. the sets of the form

(3.2) $$\{\omega: \omega(e_1) = \varepsilon_1, \dots, \omega(e_n) = \varepsilon_n\}, \; e_i \in \mathcal{E}, \; \varepsilon_i = \pm 1.$$

For the probability measure on $\mathcal{B}_{\mathcal{E}}$ we choose a product measure

(3.3) $$P_{\mathcal{E}} = \prod_{e \in \mathcal{E}} \mu_e ,$$

where μ_e is defined by

(3.4) $\mu_e\{\omega(e) = +1\} = 1 - \mu_e\{\omega(e) = -1\} = p(e)$

for some $0 \leq p(e) \leq 1$. One defines $\mathcal{B}_{\mathcal{V}}$ and $P_{\mathcal{V}}$ by replacing $\mathcal{E}$ and e by $\mathcal{V}$ and v, respectively, in (3.2) - (3.4).

Let $\omega \in \Omega_{\mathcal{E}}$. The open cluster $W(e) = W(e,\omega)$ of an edge e is the union of all edges and vertices which belong to some path $r = (v_0, e_1 = e, \dots, e_\nu, v_\nu)$ on $\mathcal{G}$, with $e_1 = e$ and all e_i passable. (We lose nothing by taking the r self-avoiding.)
For the site problem and $\omega \in \Omega_{\mathcal{V}}$ we defined the occupied cluster $W(v) = W(v,\omega)$ of a vertex v in Def. 2.7. We can of course use this same definition with "occupied" replaced by "open" to define $W(v)$ in the bond-problem. This is what we used in the introduction, but for the comparison of bond and site-problems it is convenient to have $W(e)$ available. Of course for the bond-problem

(3.5) $$W(v) = \bigcup_{\substack{e \text{ incident} \\ \text{to } v}} W(e)$$

so that there is a close relation between $W(e)$ and $W(v)$. We shall use

(3.6) $\#W(e)$ and $\#W(v)$

to denote the number of edges in $W(e)$ and the number of vertices in $W(v)$. The principal questions in percolation theory concern the distribution of $\#W$, in particular the dependence of this distribution on the parameters $p(e)$ and $p(v)$ of $P_{\mathcal{E}}$ and $P_{\mathcal{V}}$. Of special interest are the percolation probabilities

$$\theta_{\mathcal{E}}(e) := P_{\mathcal{E}}\{\#W(e) = \infty\} \quad \text{and}$$

$$\theta_{\mathcal{V}}(v) := P_{\mathcal{V}}\{\#W(v) = \infty\}.$$

The description in this section nowhere refers to the embedding of $\mathcal{G}$ in $\mathbb{R}^d$. It is therefore clear that the distribution of $\#W$ and all related quantities in percolation theory depend only on the abstract structure of $\mathcal{G}$, i.e., on $\mathcal{V}$ and $\mathcal{E}$ and the adjacency relationship. The embedding merely helps us to visualize the situation and to give economical proofs.

Before narrowing down the model further we show that a bond-percolation problem on $\mathcal{G}$ is equivalent to a site-percolation problem on $\tilde{\mathcal{G}}$, the covering graph of $\mathcal{G}$ (see Def. 2.13). For instance the distribution of $\# W(e)$ for an edge e of the simple quadratic lattice $\mathcal{G}_0$ of Ex. 2.1 (i) will be the same as that of $\# W(\tilde{v})$

when $\tilde{v}$ is the vertex of the graph $\mathcal{G}_1$ of Ex. 2.1 (ii) which corresponds to e $(\mathcal{G}_1 = \tilde{\mathcal{G}}_0)$, and when the probability measures on $\mathcal{G}_0$ and $\mathcal{G}_1$ are suitably related. In general, let $\mathcal{G}$ be a graph with covering graph $\tilde{\mathcal{G}}$. Temporarily write a tilde over the entitities introduced above to denote the corresponding entity for $\tilde{\mathcal{G}}$. (e.g. $\tilde{\Omega}_{\tilde{\mathcal{V}}}$, $\tilde{P}_{\tilde{\mathcal{V}}}$). Denote by $\tilde{v}(e)$ the vertex of $\tilde{\mathcal{G}}$ associated to the edge e of $\mathcal{G}$ (see Def. 2.13). We then have the following proposition.

<u>Proposition 3.1</u>. <u>Let</u> $\mathcal{G}$ <u>be a graph with covering graph</u> $\tilde{\mathcal{G}}$. <u>Define the map</u> $\phi : \Omega_{\mathcal{E}} \to \tilde{\Omega}_{\tilde{\mathcal{V}}}$ <u>by</u>

(3.7) $$\phi(\omega)(\tilde{v}(e)) = \omega(e) \quad , \quad e \in \mathcal{E} .$$

<u>Then</u> ϕ <u>is</u> 1-1 <u>onto</u> $\tilde{\Omega}_{\tilde{\mathcal{V}}}$, <u>and for any</u> $e \in \mathcal{E}$, $\omega \in \Omega_{\mathcal{E}}$,

(3.8) $$f \in W(e,\omega) \text{ if and only if } \tilde{v}(f) \in \tilde{W}(\tilde{v}(e) , \phi(\omega)).$$

<u>Moreover, if</u> $\tilde{P}_{\tilde{\mathcal{V}}}$ <u>is defined by</u>

(3.9) $$\tilde{P}_{\tilde{\mathcal{V}}} = \prod_{\tilde{v} \in \tilde{\mathcal{V}}} \mu_{\tilde{v}}$$

<u>with</u>

(3.10) $$\tilde{\mu}_{\tilde{v}}\{\tilde{\omega}(\tilde{v}) = +1\} = \mu_e\{\omega(e) = 1\} = p(e)$$

<u>whenever</u> $\tilde{v} = \tilde{v}(e)$, <u>then for all</u> $n \leq \infty$

(3.11) $$\tilde{P}_{\tilde{\mathcal{V}}}\{\#\tilde{W}(\tilde{v}(e)) = n\} = P_{\mathcal{E}}\{\#W(e) = n\}.$$

<u>Proof:</u> $f \in W(e,\omega)$ iff there exists a path $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ on $\mathcal{G}$ with $\omega(e_i) = 1$ and $e_1 = e$, $e_\nu = f$. For any such r let $\tilde{r} = (\tilde{v}_1,\tilde{e}_1,\dots,\tilde{v}_\nu)$ be a path with possible double points with $\tilde{v}_i = \tilde{v}(e_i)$ associated to r as in Comment 2.5(iii). Then, by (3.7) $\phi(\omega)(\tilde{v}_i) = \phi(\omega)(\tilde{v}(e_i)) = 1$ so that $f \in W(e,\omega)$ implies

$$\tilde{v}_\nu = \tilde{v}(e_\nu) = \tilde{v}(f) \in \tilde{W}(\tilde{v}_1,\phi(\omega)) = \tilde{W}(\tilde{v}(e), \phi(\omega)) .$$

The other direction of (3.8) is proved in the same way.

Now let C be a fixed union of n distinct edges of $\mathcal{G}$ containing e and such that for each edge $f \in C$ there exists a path $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ with possible double points on $\mathcal{G}$ with $e_1 = e$, $e_\nu = f$. Then $W(e,\omega) = C$ occurs iff

(3.12) $\omega(f)=1$ for all $f \in C$, but $\omega(g) = -1$ for all edges g of $\mathcal{G}$ with one endpoint in C, but g not belonging to C.

Indeed the first requirement of (3.12) says that each edge in C belongs to $W(e,\omega)$, while the second requirement says that no other edges f belong to $W(e,\omega)$, for any path from e to an edge outside C has to contain an edge outside C with one endpoint in C. Next let $\tilde{C}$ be the union of all vertices $\tilde{v}(f)$, $f \in C$, and all edges of $\tilde{\mathcal{G}}$ between any two such vertices. $\tilde{C}$ is contained in $\tilde{\mathcal{G}}$ and contains exactly the n distinct vertices $\tilde{v}(f)$, $f \in C$, including of course $\tilde{v}(e)$. Moreover $\tilde{W}(\tilde{v}(e), \tilde{\omega}) = \tilde{C}$ iff

(3.13) $\tilde{\omega}(\tilde{w}) = 1$ for all $\tilde{w} \in \tilde{C}$, but $\tilde{\omega}(\tilde{u}) = -1$ for all vertices $\tilde{u}$ of $\tilde{\mathcal{G}}$ adjacent to a vertex in $\tilde{C}$, but not belonging to $\tilde{C}$

One easily sees that g has an endpoint in C but does not belong to C iff $\tilde{v}(g)$ is adjacent to some vertex of $\tilde{C}$, but $\tilde{v}(g) \notin \tilde{C}$. From this it is easy to see that

$$(3.14) \qquad P_{\mathcal{E}} \{W(e) = C\} = \tilde{P}_{\tilde{\mathcal{V}}} \{\tilde{W}(\tilde{v}(e)) = \tilde{C}\}$$

if one takes $\tilde{P}_{\tilde{\mathcal{V}}}$ as in (3.9), (3.10) . But

$$(3.15) \qquad \{\#W(e) = n\} = \bigcup_{\#C=n} \{W(e) = C\}$$

with the union in the right hand side of (3.15) being over all C of the type considered above and containing n edges. Similarly

$$(3.16) \qquad \{\#\tilde{W}(\tilde{v}(e)) = n\} = \bigcup_{\#\tilde{C}=n} \{\tilde{W}(\tilde{v}(e)) = \tilde{C}\}$$

and each $\tilde{C}$ in the right hand side of (3.16) is the image of a unique C in the right hand side of (3.15). The last statement is easily verified by means of Comment 2.5(iii). (3.11) now follows from (3.14)-(3.16). □

Because of Prop. 3.1 we shall restrict ourselves henceforth to site-percolation. The subscripts $\mathcal{V}$ used in this section therefore become superfluous and will be dropped from now on. We remark that we cannot use a similar procedure to translate a site-percolation problem on every graph $\mathcal{G}$ to a bond-percolation problem on another graph, because $\mathcal{G}$ may not be a covering graph of any other graph. (If $\mathcal{G} = \tilde{\mathcal{H}}$ for some graph $\mathcal{H}$, and $\mathcal{H}$ has any vertex with three distinct edges e_1, e_2, e_3 incident to it, then $\tilde{v}(e_1)$, $\tilde{v}(e_2)$ and $\tilde{v}(e_3)$ are the vertices of a "triangle" in $\mathcal{G}$. Thus the graph $\mathcal{G}_0$ of Ex. 2.1 (i) - which has no triangles - is not a covering graph.) On the other hand, there seems to be no way to go from site-percolation on $\mathcal{G}$ to bond-percolation on $\tilde{\mathcal{G}}$.

3.2. Periodic site-percolation.

Let $\mathcal{G}$ be a periodic graph, imbedded in $\mathbb{R}^d$, with vertex set $\mathcal{V}$ (see Def. 2.1). We consider a periodic partition of $\mathcal{V}$ into λ sets $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$, i.e., we assume

(3.17) $$\mathcal{V}_i \cap \mathcal{V}_j = \phi \ , \ i \neq j \ , \ \mathcal{V} = \bigcup_{i=1}^{\lambda} \mathcal{V}_i \ ,$$

and (with $\xi_1,\dots,\xi_d$ the coordinate vectors of $\mathbb{R}^d$)

(3.18) $$v \in \mathcal{V}_i \quad \text{iff} \quad v + \sum_{j=1}^{d} k_j\xi_j \in \mathcal{V}_i \ ,$$

$$1 \leq i \leq \lambda \ , \ k_j \in \mathbb{Z} \, .$$

(In typical examples the $\mathcal{V}_i$ will only have periods which are multiples of $\xi_1,\dots,\xi_d$ and one has to change scale to obtain (3.18); see Ex. 3.2(i) below). We take, as in Sect 3.1

(3.19) $$\Omega = \prod_{\mathcal{V}}\{-1,+1\}$$

and $\mathcal{B}$ the σ-field generated by the cylinder sets in Ω. We shall restrict ourselves to probability measures on $\mathcal{B}$ which are specified by λ parameters as follows: Let

(3.20) $$\mathcal{P}_\lambda = [0,1]^\lambda$$

and

(3.21) $$p = (p(1),\dots,p(\lambda)) \in \mathcal{P}_\lambda$$

Then take

(3.22) $$P_p = \prod_{v\in\mathcal{V}} \mu_v \ ,$$

where

(3.23) $$\mu_v\{\omega(v)=1\} = 1-\mu_v\{\omega(v)=-1\} = p(i) \quad \text{if} \quad v \in \mathcal{V}_i, \ i\leq i\leq\lambda \ .$$

<u>A probability measure of this form will be called a (λ-parameter) periodic probability measure. Henceforth we shall consider only periodic probability measures on periodic graphs.</u> E_p will denote expectation with respect to P_p.

<u>Examples.</u>

(i) Let $\mathcal{G}_0$ be the periodic graph of Ex. 2.1(i), the simple quadratic lattice. Take $\lambda = 2$, $\mathcal{V}_1 = \{(i_1,i_2): i_1+i_2 \text{ is even}\}$, $\mathcal{V}_2 = \{(i_1,i_2): i_1+i_2 \text{ is odd}\}$. As it stands, this does not satisfy (3.18). However, we only have to make a change of scale to put the example in periodic form. We change the imbedding so that the vertex originally at (i_1,i_2) is now at $(\frac{i_1}{2}, \frac{i_2}{2})$, and similarly "multiply the edges by a factor $\frac{1}{2}$" . $(\frac{i_1}{2}, \frac{i_2}{2})$ and $(\frac{j_1}{2}, \frac{j_2}{2})$ are neighbors iff (2.6) holds. $\mathcal{V}_1$ now becomes $\{(\frac{i_1}{2}, \frac{i_2}{2}) : i_1+i_2$ is even$\}$ and similarly for $\mathcal{V}_2$.

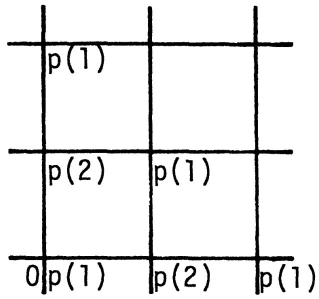

Figure 3.1 Two-parameter site-percolation on $\mathbb{Z}^2$. The p-value next to a vertex gives the probability of being occupied for that vertex.

(ii) We describe this example as a bond-problem, because the transcription to a site-problem on the covering graph is more complicated. In this example we allow three parameters. For $\mathcal{G}$ we take the triangular lattice of Ex. 2.1(iii). We now consider the partition of its <u>bonds</u> into the three sets

(3.24) $\quad \mathcal{E}_j$ = {bonds along the lines under an angle $(j-1)\frac{2\pi}{3}$ with the first coordinate axis}, j=1,2,3,

and take each bond in $\mathcal{E}_j$ open with probability p(j). The description in (3.24) presupposes that $\mathcal{G}$ is imbedded in $\mathbb{R}^2$ as in Fig. 2.4

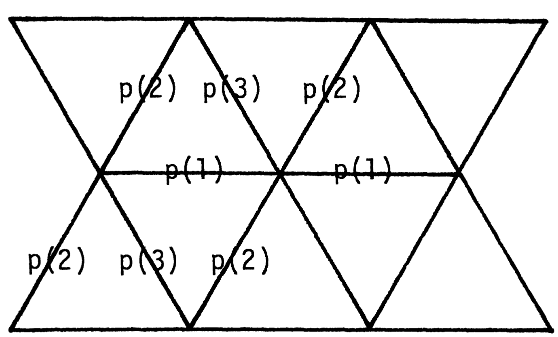

Figure 3.2 The p-value next to an edge gives the probability of that edge being passable.

To transcribe this to a periodic site-problem, we have to assign probability $p(j)$ of being occupied to a vertex of $\tilde{\mathcal{G}}$ corresponding to an edge e in $\mathcal{E}_j$. We also have to change scale as in the description of $\mathcal{T}$ in Ex. 2.1 (iii) to obtain a periodic problem. ///

Define $W(v) = W(v,\omega)$ as in Def. 2.7 with "v occupied" being interpreted as "$\omega(v) = +1$", and set

$$(3.25) \qquad \theta(p,v) = P_p\{\#W(v) = \infty\} .$$

The λ parameter periodic site-percolation problem is to determine the percolative region in $\mathcal{P}_\lambda$, i.e., to determine the set

$$(3.26) \qquad \{p \in \mathcal{P}_\lambda : \theta(p,v) > 0 \text{ for some } v\} .$$

If $p(i) > 0$ for $i \le i \le \lambda$ then $\theta(p,v) > 0$ for some v iff $\theta(p,v) > 0$ for all v by the FKG inequality (see Broadbent and Hammersley (1957) and Sect. 4.1. below). Therefore the intersection of the set (3.26) with $\{p: p(i) > 0 \text{ for } 1 \le i \le \lambda\}$ is independent of v ; it equals the set

$$(3.27) \qquad \{p \in \mathcal{P}_\lambda : p(i) > 0 \text{ for } 1 \le i \le \lambda \text{ and } \theta(p,v) > 0 \text{ for all } v\}.$$

In the next section we formulate our principal result describing the percolative region, while Sect. 3.4 applies this theorem to give explicit answers in a number of examples. These answers had all been conjectured already in Sykes and Essam (1964).

3.3. Crossing probabilities and the principal theorem on percolative regions.

Let $\mathcal{G}$ be a graph imbedded in $\mathbb{R}^d$ which satisfies (2.1)-(2.5). We consider blocks B in $\mathbb{R}^d$ of the form

$$(3.28) \qquad B = \prod_1^d [a_i,b_i] = \{x=(x(1),\dots,x(d)): a_i \le x(i) \le b_i, 1 \le i \le d\}$$

Def. 1. An i-crossing (on $\mathcal{G}$) of B is a path $(v_0,e_1,\dots,e_\nu,v_\nu)$ (on $\mathcal{G}$) which satisfies[1)]

$$(3.29) \qquad (v_1,e_2,\dots,e_{\nu-1},v_{\nu-1}) \text{ is contained in } \mathring{B} = (a_1,b_1) \times \dots \times (a_d,b_d)$$

1) We use standard interval notation for segments of edges. E.g. in (3.30) $(\zeta_1,v_1]$ denotes the piece of e_1 between ζ_1 and v_1 excluding ζ_1 but including v_1 . Similarly for the segment $[v_{\nu-1},\zeta_\nu)$ of e_ν in (3.31) .

(3.30) e_1 intersects the face $\{x(i) = a_i\} \cap B =$

$[a_1,b_1]\times\ldots\times[a_{i-1},b_{i-1}]\times\{a_i\}\times[a_{i+1},b_{i+1}]\times\ldots\times[a_d,b_d]$

of B in some point ζ_1 such that $(\zeta_1,v_1] \subset \mathring{B}$.

and

(3.31) e_ν intersects the face $\{x(i) = b_i\} \cap B$ of B in some point ζ_ν such that $[v_{\nu-1},\zeta_\nu) \subset \mathring{B}$.

Comments .

(i) Note that (3.29)-(3.31) require all but the first and final edge of an i-crossing $(v_0,e_1,\ldots,e_\nu,v_\nu)$ of B, as well as the segments $(\zeta_1,v_1]$ and $[v_{\nu-1},\zeta_\nu)$ of the first and final edge to lie in the interior of B. When $d = 2$, i.e. B is a rectangle in the plane, then we shall call a 1-crossing (2-crossing) a <u>horizontal (vertical) crossing</u> of B. In this case the continuous curve made up from $[\zeta_1,v_1]$, $e_2,\ldots,e_{\nu-1}$ and $[v_{\nu-1},\zeta_{\nu-1}]$ is a crosscut of $\mathring{B}$ (in the terminology of Newman (1951), Ch. V.11.). Finally note that the initial and final point v_0 and v_ν of a crossing of B can lie in $\mathring{B}$ or in Fr (B) or outside B.

(ii) An i-crossing r of B is minimal in the sense that no subpath of the crossing with fewer edges than r is still an i-crossing. One does, however, have the following obvious monotonicity property. If $[a_i',b_i'] \subset [a_i,b_i]$ but $[a_j,b_j] \subset [a_j',b_j']$ for $j \neq i$, then an i-crossing $(v_0,e_1,\ldots,e_\nu,v_\nu)$ of $B = \Pi[a_j,b_j]$ contains a subpath $(v_\alpha,e_{\alpha+1},\ldots,e_\beta,v_\beta)$ which is an i-crossing of $B' = \Pi[a_j',b_j']$.

<u>Def. 2</u>. An i-crossing $(v_0,e_1,\ldots,e_\nu,v_\nu)$ of B is called an <u>occupied (vacant) i-crossing</u> if all its vertices are occupied (vacant).

Comments .

(iii) When we shall use vacant crossings we shall usually be dealing with a matching pair of graphs $\mathcal{G}$ and $\mathcal{G}^*$. We shall then be interested in occupied crossings on $\mathcal{G}$ and vacant crossings on $\mathcal{G}^*$. ///

Now let P_p be a λ-parameter periodic probability measure, as in Sect. 3.2. Especially important for us will be the probability that there exists an i-crossing of a block with the "lower left" corner at the origin. Formally we define these as follows.

<u>Def. 3</u>. The <u>crossing probability in the i-th direction of</u> $[0,n_1] \times \ldots \times [0,n_d]$ (on $\mathcal{G}$) is

(3.32) $\sigma(\bar{n};i,p) = \sigma(\bar{n};i,p,\mathcal{G}) = P_p\{\exists$ an occupied i-crossing on $\mathcal{G}$ of $[0,n_1] \times \dots \times [0,n_d]\}$.

The analogous quantity for vacant crossings on $\mathcal{G}^*$ will be written as

(3.33) $\sigma^*(\bar{n};i,p) = \sigma^*(\bar{n};i,p,\mathcal{G}) = \sigma(\bar{n};i,\bar{1}-p,\mathcal{G}^*) = P_p\{\exists$ a vacant i-crossing on $\mathcal{G}^*$ of $[0,n_1] \times \dots \times [0,n_d]\}$

($\bar{n}$ here stands for $(n_1,\dots,n_d)$.)

<u>Comments</u>.

(iv) In (3.33) $\bar{1}-p$ stands for the λ-vector $(1-p(1),1-p(2),\dots,1-p(\lambda))$, while $(\mathcal{G},\mathcal{G}^*)$ is a matching pair, based on $(\mathcal{M},\mathcal{F})$ say. Recall that $\mathcal{M},\mathcal{G}$ and $\mathcal{G}^*$ have the same vertex set in this case (Comment 2.2 (iv)). Thus P_p as defined by (3.21)-(3.23) is simultaneously a probability measure on the occupancy configurations on $\mathcal{G}_1$, on $\mathcal{G}^*$ and on $\mathcal{M}$. The second equality in (3.33) is immediate from

(3.34) $P_p\{v \text{ is vacant}\} = 1-p(i) = P_{\bar{1}-p}\{v \text{ is occupied}\}, \quad v \in \mathcal{V}_i.$

(see (3.22), (3.23)).

(v) It is immediate from Def. 3.1, 3.2 and Comment 3.3(ii) that $\sigma(\bar{n};i,p)$ is decreasing in n_i but increasing in each n_j with $j \neq i$. ///

The remainder of this section gives the formulation of our principal theorems on the percolative region. These deal only with graphs imbedded in the plane. $(\mathcal{G}, \mathcal{G}^*)$ will be a matching pair of periodic graphs imbedded in $\mathbb{R}^2$, and P_p will be a λ-parameter probability measure. $W^*(v) = W^*(v,\omega)$ will denote the vacant cluster of v on $\mathcal{G}$, i.e.,the union of all edges and vertices of $\mathcal{G}^*$ which belong to a vacant path on $\mathcal{G}^*$ with initial point v. The following conditions A and B will be used. They are viewed as conditions on the parameter point p_0 for fixed $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$. Condition A relates the probabilities of an occupied crossing on $\mathcal{G}$ with those of a vacant crossing on $\mathcal{G}^*$. Condition B is a relation between horizontal crossings (i.e., crossings in the 1-direction) with vertical crossings (i.e., crossings in the 2-direction).

<u>Condition A</u>. There exists a $0 < \delta \le \frac{1}{2}$, an integer n_0 and

vectors[1] $\bar{\rho} = (\rho_1,\rho_2), \bar{\rho}^* = (\rho_1^*,\rho_2^*)$ such that for $i = 1$ or $i = 2$

$$\sigma(\bar{n};\, i, p_0) \geq \frac{1}{2} \text{ implies } \sigma^*(\bar{n}-\bar{\rho}; i, p_0) \geq \delta , \quad \text{whenever } n_1, n_2 \geq n_0 , \tag{3.35}$$

and also for $j=1$ or $j=2$

$$\sigma^*(\bar{n}; j, p_0) \geq \frac{1}{2} \text{ implies } \sigma(\bar{n}-\bar{\rho}^*; j, p_0) \geq \delta \quad \text{whenever } n_1, n_2 \geq n_0 . \tag{3.36}$$

<u>Condition B</u>. There exist numbers $\delta > 0$, $0 < a_j, b_j < \infty$, $j = 1,2$, and sequences $\{\bar{n}_\ell = (n_{\ell 1}, n_{\ell 2})\}_{\ell \geq 1}, \{\bar{m}_\ell = (m_{\ell 1}, m_{\ell 2})\}_{\ell \geq 1}$ such that

$$n_{\ell j} \to \infty , \; m_{\ell j} \to \infty \text{ as } \ell \to \infty , \; j = 1,2 \tag{3.37}$$

and

$$\sigma(\bar{n}_\ell;\, 1, p_0) \geq \delta , \; \sigma((a_1 n_{\ell 1}, a_2 n_{\ell 2});\, 2, p_0) \geq \delta \tag{3.38}$$

$$\sigma^*(\bar{m}_\ell; 1, p_0) \geq \delta , \; \sigma^*((b_1 m_{\ell 1}, b_2 m_{\ell 2});\, 2, p_0) \geq \delta . \tag{3.39}$$

One more definition and a bit of notation.

<u>Def. 4</u> We call the line $L : x(1) = a$ or $x(2) = a$ an <u>axis of symmetry for the partition</u> $\mathcal{V}_1, \ldots, \mathcal{V}_\lambda$ of the vertices of $\mathcal{G}$ if each $\mathcal{V}_i$ is invariant under reflection in the line L.

<u>Comment</u>.

(vi) If P_p is given by (3.22), (3.23) and $x(1) = a$ is an axis of symmetry for $\mathcal{G}$ and for $\mathcal{V}_1, \ldots, \mathcal{V}_\lambda$ then for $v = (v(1), v(2))$

$$P_p\{v = (v(1),v(2)) \text{ is occupied}\} = P_p\{(2a-v(1),v(2)) \text{ is occupied}\} \tag{3.40}$$

for any $p \in \mathcal{P}_\lambda$. Similarly if $x(2) = a$ is an axis of symmetry for $\mathcal{G}$ and $\mathcal{V}_1, \ldots, \mathcal{V}_\lambda$. ///

When dealing with λ-parameter problems $\bar{0}(\bar{1})$ will denote the λ-vector all of whose components equal zero (one). For $p \in \mathcal{P}_\lambda$, and real t, tp has components $tp(1), \ldots, tp(\lambda)$. Also, for $p', p'' \in \mathcal{P}_\lambda$

$$p' << p'' \text{ means } p'(i) < p''(i) \; , \; i \leq i \leq \lambda . \tag{3.41}$$

Unfortunately the following two theorems have a forbidding appearance. Nevertheless they allow the determination of the percolative

1) The ρ_i and ρ_i^* can take negative values.

region in several examples, as we demonstrate in the next section.

Theorem 3.1. Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic graphs imbedded in $\mathbb{R}^2$ and $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ a periodic partition of the vertices of $\mathcal{G}$ such that one of the coordinate axis is an axis of symmetry for $\mathcal{G}$, $\mathcal{G}^*$ and the partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$. Let $p_0 \in \mathcal{P}_\lambda$ be such that

(3.42) $$\overline{0} << p_0 << \overline{1}$$

and such that Condition A or Condition B is satisfied. Then

(i) for all vertices v of $\mathcal{G}$ (and hence of $\mathcal{G}^*$)

(3.43) $$P_{p_0}\{\#W(v) = \infty\} = P_{p_0}\{\#W^*(v) = \infty\} = 0$$

but

(3.44) $$E_{p_0}\{\#W(v)\} = E_{p_0}\{\#W^*(v)\} = \infty .$$

Also, for every square $S_N = \{(x_1,x_2) : |x_1| \leq N, |x_2| \leq N\}$

(3.45) $$P_{p_0}\{\exists \text{ an occupied circuit on } \mathcal{G} \text{ surrounding } S_N \text{ and } \exists \text{ a vacant circuit on } \mathcal{G}^* \text{ surrounding } S_N\} = 1 .$$

(ii) for any $p' << p_0$

(3.46) $$P_{p'}\{\#W(v) = \infty\} = 0 , \; P_{p'}\{\#W^*(v) = \infty\} > 0$$

and

(3.47) $$P_{p'}\{\exists \text{ exactly one infinite vacant cluster on } \mathcal{G}^*\} = 1$$

and

(3.48) $$E_{p'}\{\#W(v)\} < \infty .$$

(iii) for any $p'' >> p_0$

(3.49) $$P_{p''}\{\#W(v) = \infty\} > 0 , \; P_{p''}\{\#W^*(v) = \infty\} = 0$$

and

(3.50) $$P_{p''}\{\exists \text{ exactly one infinite occupied cluster on } \mathcal{G}\} = 1$$

and

(3.51) $$E_{p''}\{\#W^*(v)\} < \infty .$$

Theorem 3.2. Let $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ be as in Theorem 3.1. Assume there exist constants $0 < a_j,\dots,d_j < \infty$, $j = 1,2,3$, and for each $p \in \mathcal{P}_\lambda$ with $\overline{0} << p << \overline{1}$ a function $h : (0,1] \to (0,1]$ and an n_0 (h and n_0 may depend on p) such that for $n \geq n_0$ and $0 < x \leq 1$

(3.52) $\sigma((n,a_1 n);1,p) \geq x$ implies $\sigma((a_2 n,a_3 n);2,p) \geq h(x) > 0$

(3.53) $\sigma((n,b_1 n);2,p) \geq x$ implies $\sigma((b_2 n,b_3 n);1,p) \geq h(x) > 0$,

(3.54) $\sigma^*((n,c_1 n);1,p) \geq x$ implies $\sigma^*((c_2 n,c_3 n);2,p) \geq h(x) > 0$,

and

(3.55) $\sigma^*((n,d_1 n);2,p) \geq x$ implies $\sigma^*((d_2 n,d_3 n);1,p) \geq h(x) > 0$.

For $p_1 \in \mathcal{P}_\lambda$ choose

(3.56) $t_0 = \inf\{t \geq 0: tp_1 \in \mathcal{P}_\lambda$, $\limsup \sigma((n,a_1 n);1,tp_1) > 0$

or $\limsup_{n\to\infty} \sigma((n,b_1 n); 2,tp_1) > 0\}$

provided the set in the right hand side of (3.56) is nonempty. If

$$\bar{0} << p_0 := t_0 p_1 << \bar{1} ,$$

then condition B holds for p_0 , and consequently also (3.43) - (3.51) .

The proof of these theorems will be given in Ch. 7 after the necessary machinery has been developed.

In all examples of the next section the following corollary applies. Let $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{V}_1,\ldots, \mathcal{V}_\lambda$ be as in Theorem 3.1. Set

(3.57) $\mathcal{S} = \{p_0 \in \mathcal{P}_\lambda : \bar{0} << p_0 << \bar{1}$ and Condition A or Condition B holds for $p_0\}$

and

(3.58) $\mathcal{P}_- = \{p' \in \mathcal{P}_\lambda: p' << p_0$ for some $p_0 \in \mathcal{S}\}$,

$\mathcal{P}_+ = \{p'' \in \mathcal{P}_\lambda: p'' >> p_0$ for some $p_0 \in \mathcal{S}\}$.

Corollary 3.1. Let $(\mathcal{G}, \mathcal{G}^*)$ and $\mathcal{V}_1,\ldots, \mathcal{V}_\lambda$ be as in Theorem 3.1. If

(3.59) $$(0,1)^\lambda \subset \mathcal{P}_- \cup \mathcal{S} \cup \mathcal{P}_+$$

then the percolative regions for $\mathcal{G}$ and $\mathcal{G}^*$ in $(0,1)^\lambda$ are $\mathcal{P}_+$ and $\mathcal{P}_-$, respectively (i.e., for $\bar{0} << p << \bar{1}$ infinite occupied clusters on $\mathcal{G}$ (infinite vacant clusters on $\mathcal{G}^*$) occur iff $p \in \mathcal{P}_+(\mathcal{P}_-)$.

It is reasonable to call $\mathcal{S}$ the critical surface in the cases where Cor. 1 applies.

3.4 Critical probabilities. Applications of the principal theorems.

The FKG inequality implies (see Sect. 4.1) that if $\mathcal{G}$ is connected, and if

(3.60) $P_p\{v \text{ is occupied}\} > 0$ for all vertices of $\mathcal{G}$,

then $\theta(p,v) > 0$ for some v iff $\theta(p,v) > 0$ for all v. Also $E_p\{\#W(v)\} = \infty$ for some v iff this holds for all v (see Sect. 4.1). For one-parameter problems with

(3.61) $P_p\{v \text{ is occupied}\} = \mu_v\{\omega(v) = 1\} = p$

for all vertices v of a connected graph $\mathcal{G}$ we can therefore define the <u>critical probabilities</u>

(3.62) $p_H = p_H(\mathcal{G}) = \sup\{p \in [0,1] : \theta(p,v) = 0\}$,

(3.63) $p_T = p_T(\mathcal{G}) = \sup\{p \in [0,1] : E_p\{\#W(v)\} < \infty\}$,

and these numbers are independent of the choice of v. By definition

$$E_p\{\#W(v)\} \geq \theta(p,v) \cdot \infty$$

so that $E_p\{\#W(v)\} = \infty$ for $p > p_H$. Therefore one always has

(3.64) $$p_T \leq p_H .$$

For periodic graphs $\mathcal{G}$ imbedded in $\mathbb{R}^d$ we define a third critical probability which is a slight modification of one introduced by Seymour and Welsh (1978); see also Russo (1978).

(3.65) $$p_S := \sup\{p \in [0,1] : \lim_{n\to\infty} \sigma((3n,3n,\ldots 3n,n,3n,\ldots 3n);i,p) = 0 \quad , \quad 1 \leq i \leq d\} .$$

where the one component equal to n in $\sigma((3n,\ldots,n,\ldots,3n);i,p)$ in (3.65) is the i-th component. It will be a consequence of Theorem 5.1 that for any periodic graph $\mathcal{G}$ imbedded in $\mathbb{R}^d$

(3.66) $$p_T = p_S .$$

In some cases Corollary 3.1 can be used to show that

$$p_T = p_S = p_H ,$$

and in a small class of examples one can even calculate the common value of these critical probabilities. This is demonstrated in the applications below. Again all these applications are for graphs imbedded in the plane.

<u>Applications</u>.

(i) <u>Triangulated graphs</u>. Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^2$ such that one of the coordinate axes is a symmetry axis and such that all faces of $\mathcal{G}$ are triangles. Let P_p be the one-parameter probability measure defined by (3.22) and (3.61). In each problem of this form

(3.67) $$p_T = p_S = p_H = \frac{1}{2} .$$

This applies for instance in the site-problem on the triangular lattice of Ex. 2.1(iii) or the centered quadratic lattice of Ex. 2.2 (iii).

It is interesting to observe that one may "decorate" the faces of $\mathcal{G}$ almost arbitrarily without affecting (3.67). That is, if F is a face of $\mathcal{G}$ we may add a number of vertices and edges inside F. The addition of these vertices and edges does not increase $\theta(p,v)$. Indeed, any occupied path entering and leaving $\bar{F}$ has to do so at two vertices v_1 and v_2 on the perimeter of F. But then v_1 and v_2 are occupied and connected by an edge of $\mathcal{G}$, and hence the piece of the path in $\bar{F}$ between v_1 and v_2 can be replaced by the edge between v_1 and v_2. We can make such a change in every face; the decorations of different faces don't have to have any relation to each other, and the resulting graph does not have to be periodic or planar. Nevertheless it will have the same value of $\theta(p,v)$ for $v \in \mathcal{G}$ and hence also $p_H = \frac{1}{2}$. If the number of added vertices in any face is uniformly bounded, then a slight extension of the above argument shows that also $p_T = p_S = \frac{1}{2}$ remains true for the decorated graph.

Van den Berg (1981), Fig. 1, shows an interesting example of a graph $\mathcal{G}$ which has all the properties required above, except for the periodicity, but with $p_T = p_H = 1$. This illustrates how crucial periodicity is.

Proof of (3.67): $\mathcal{G}$ is a periodic mosaic and since all faces are already close-packed, we can take $\mathcal{G}^* = \mathcal{G}$. $(\mathcal{G}, \mathcal{G}^*)$ is the matching pair based on $(\mathcal{G}, \emptyset)$; see Ex. 2.2 (iii). $\mathcal{G}$ is self-matching and Condition A holds trivially for $p_0 = \frac{1}{2}$. Indeed

$$P_{\frac{1}{2}}\{v \text{ is occupied}\} = P_{\frac{1}{2}}\{v \text{ is vacant}\} = \frac{1}{2} ,$$

and since $\mathcal{G} = \mathcal{G}^*$ this gives

(3.68) $$\sigma^*(\bar{n};i,\tfrac{1}{2}) = \sigma(\bar{n};i,1-\tfrac{1}{2},\mathcal{G}^*) = \sigma(\bar{n};i,\tfrac{1}{2},\mathcal{G}) .$$

Clearly (3.68) implies (3.35) and (3.36). Thus, by (3.43), (3.46) and (3.49) percolation occurs under P_p iff $p > \frac{1}{2}$. Also,

$E_p\{\#W(v)\} < \infty$ iff $p < \frac{1}{2}$. Thus $p_H = p_T = \frac{1}{2}$ and (3.67) now follows from (3.66).

(ii) <u>Bond percolation on $\mathbb{Z}^2$ and further self-matching problems.</u> In the first application we considered a one-parameter problem with $\mathcal{G} = \mathcal{G}^*$. Here we consider a two-parameter problem for a matching pair of periodic graphs $(\mathcal{G}, \mathcal{G}^*)$ with $\mathcal{G}^*$ a translate of $\mathcal{G}$. Assume that

(3.69) $$\mathcal{G}^* = \mathcal{G} + \gamma$$

for some vector $\gamma = (\gamma(1),\gamma(2))$. In other words, $\mathcal{G}$ and $\mathcal{G}^*$ are imbedded in $\mathbb{R}^2$ such that $v(e)$ is a vertex (edge) of $\mathcal{G}$ iff $v + \gamma$ $(e + \gamma)$ is a vertex (edge) of $\mathcal{G}^*$. Assume also that the vertex set $\mathcal{V}$ is partitioned into two periodic classes $\mathcal{V}_1$, $\mathcal{V}_2$ which satisfy

(3.70) $$\mathcal{V}_2 = \mathcal{V}_1 + \gamma ,$$

and that one of the coordinate axes is an axis of symmetry for $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{V}_1$, $\mathcal{V}_2$. If $p = (p(1),p(2))$ satisfies

(3.71) $$p(1) + p(2) = 1 \quad , \quad 0 < p(i) < 1 ,$$

then it is again easy to verify Condition A (see below). Hence <u>(3.71) gives the critical surface in this situation</u>, and percolation occurs on $\mathcal{G}$ under P_p with $p << 1$ iff $p(1) + p(2) > 1$. The restriction of p to the line $p(1) = p(2)$ gives the one-parameter problem, and we see from (3.71) that the critical probabilities are again given by (3.67) in a one-parameter problem on a $\mathcal{G}$ which satisfies (3.69) ((3.70) will not even be needed for the one-parameter problem, since (3.72) below automatically holds at $p = (\frac{1}{2},\frac{1}{2})$.).

The most classical example of this kind is bond-percolation on $\mathbb{Z}^2$ with

$$P\{e \text{ is passable}\} = \begin{cases} p(1) & \text{if } e \text{ is a horizontal edge} \\ p(2) & \text{if } e \text{ is a vertical edge.} \end{cases}$$

By Prop. 3.1 this is equivalent to site-percolation on the graph $\mathcal{G}_1$ of Ex. 2.1 (ii) with

$$\mathcal{V}_1 = \{(i_1 + \tfrac{1}{2}, i_2) : i_1,i_2 \in \mathbb{Z}\} ,$$

$$\mathcal{V}_2 = \{(i_1,i_2 + \tfrac{1}{2}) : i_1,i_2 \in \mathbb{Z}\} .$$

(See also Ex. 2.5 (ii).) To see that this fits in the above framework we take for $\mathcal{M}_1$ the mosaic with vertex set $\mathcal{V}_1 \cup \mathcal{V}_2$ and an edge between the vertices $v = (v(1),v(2))$ and $w = (w(1),w(2))$ iff (2.10) holds. For $\mathcal{F}_1$ we take the faces of $\mathcal{M}_1$ (which are tilted squares, see Fig. 3.3 below) which contain a point (i_1,i_2), with integral i_1, i_2. $\mathcal{F}_1^*$ will consist of those faces which do not contain a point (i_1,i_2) with integral i_1,i_2. Finally $\mathcal{G}_1^*$ is the graph with vertex set $\mathcal{V}_1 \cup \mathcal{V}_2$ and $v = (v(1),v(2))$, $w = (w(1),w(2))$ adjacent iff either (2.10) holds or

$$v(1) = w(1) \in \mathbb{Z} + \frac{1}{2}\ , \ v(2),w(2) \in \mathbb{Z}\ , \ |v(2)-w(2)| = 1$$

or

$$v(1),w(1) \in \mathbb{Z}\ , |v(1)-w(1)| = 1,\ v(2) = w(2) \in \mathbb{Z} + \frac{1}{2}\ .$$

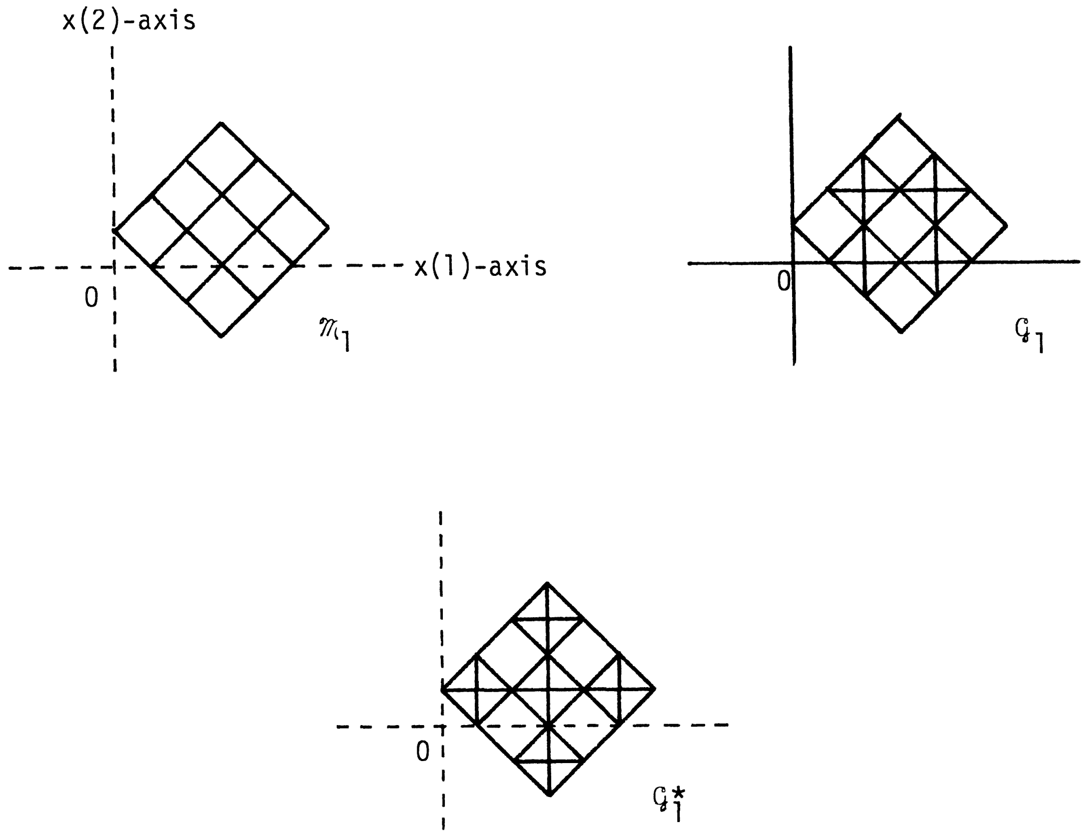

Figure 3.3

One easily checks that $(\mathcal{G}_1, \mathcal{G}_1^*)$ is the matching pair based on $(\mathcal{M}_1, \mathcal{F}_1)$ and that

$$\mathcal{G}_1^* = \mathcal{G}_1 + (\tfrac{1}{2}, \tfrac{1}{2}), \quad \mathcal{V}_2 = \mathcal{V}_1 + (\tfrac{1}{2}, \tfrac{1}{2}).$$

Thus (3.69) and (3.70) hold in this example and (3.71) is the critical surface. A generalization of this result for a mixed percolation model in which bond, sites and faces are random is given by Wierman (1982b).

Another example is $\mathcal{G} = \mathcal{G}^* = \mathcal{T}$, the triangular lattice $\mathcal{T}$ of Ex. 2.1 (iii) with

$$\mathcal{V}_1 = \{(i_1, i_2) : i_1, i_2 \in \mathbb{Z}\},$$

$$\mathcal{V}_2 = \{(i_1 + \tfrac{1}{2}, i_2 + \tfrac{1}{2}) : i_1, i_2 \in \mathbb{Z}\}.$$

Again (3.69) and (3.70) hold with $\gamma = (\frac{1}{2}, \frac{1}{2})$, and the critical surface is given by (3.71).

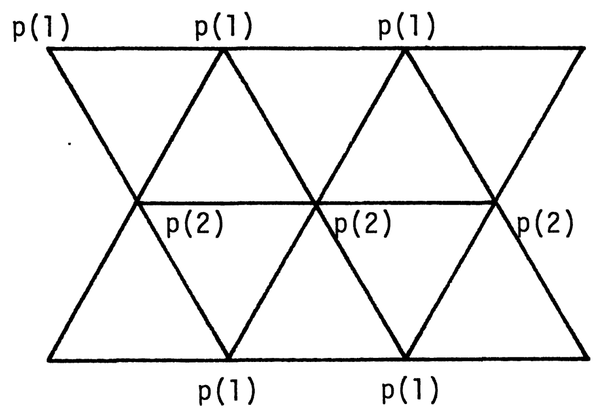

Figure 3.4 The p-value next to a vertex gives the probability of being occupied for that vertex.

Verification of Condition A. Since $\mathcal{G}$ and $\mathcal{G}^*$ have the same vertex set (Comment 2.2 (iv)).

$$\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V} = \text{vertex set of } \mathcal{G}^* = \mathcal{V} + \gamma = (\mathcal{V}_1 + \gamma) \cup (\mathcal{V}_2 + \gamma)$$

$$= \mathcal{V}_2 \cup (\mathcal{V}_2 + \gamma)$$

by (3.69) and (3.70). But $\mathcal{V}_1+\gamma$ and $\mathcal{V}_2+\gamma$ are disjoint, and the same holds for $\mathcal{V}_1$ and $\mathcal{V}_2$ (see (3.17)). Thus, in addition to (3.70)

$$\mathcal{V}_1 = \mathcal{V}_2 + \gamma \quad .$$

Therefore, if $v \in \mathcal{V}_1$, $v+\gamma \in \mathcal{V}_2$ and for p_0 satisfying (3.71)

$$P_{p_0}\{v+\gamma \text{ is vacant}\} = 1 - P_{p_0}\{v+\gamma \text{ is occupied}\} = 1 - p_0(2)$$
$$= p_0(1) = P_{p_0}\{v \text{ is occupied}\}.$$

Similarly for $v \in \mathcal{V}_2$, so that for all v

(3.72) $$P_{p_0}\{v \text{ is occupied}\} = P_{p_0}\{v+\gamma \text{ is vacant}\}, \text{ and}$$

$$P_{p_0}\{v \text{ is vacant}\} = P_{p_0}\{v+\gamma \text{ is occupied}\}.$$

Consequently the distribution of the set of occupied vertices of $\mathcal{G}$ equals the distribution of the set of vacant vertices on $\mathcal{G}+\gamma = \mathcal{G}^*$. Therefore

(3.73) $$\sigma^*(\bar{n}-\bar{\rho};1,p_0) = P_{p_0}\{\exists \text{ vacant horizontal crossing on } \mathcal{G}^* \text{ of } [0,n_1-\rho_1]\times[0,n_2-\rho_2]\}$$

$$= P_{p_0}\{\exists \text{ occupied horizontal crossing on } \mathcal{G} \text{ of } [-\gamma_1,n_1-\rho_1-\gamma_1]\times[-\gamma_2,n_2-\rho_2-\gamma_2] \quad .$$

By means of the monotonicity properties of σ given in Comment 3.3 (v) we see that for

(3.74) $$\rho_1 \geq 1 \ , \quad \rho_2 \leq -1$$

the last member of (3.73) is at least[1)]

$$P_{p_0}\{\exists \text{ occupied horizontal crossing on } \mathcal{G} \text{ of }$$
$$[-\gamma_1+\lceil \gamma_1 \rceil, n_1-\rho_1-\gamma_1+\lceil \gamma_1 \rceil]\times[-\gamma_2+\lfloor \gamma_2 \rfloor, n_2-\rho_2-\gamma_2+\lfloor \gamma_2 \rfloor]\}$$
$$\geq \sigma(\bar{n};1,p_0).$$

---

1) $\lfloor \gamma \rfloor$ denotes the largest integer $\leq \gamma$ and $\lceil \gamma \rceil$ the smallest integer $\geq \gamma$.

Thus, for any ρ which satisfies (3.74), (3.35) holds with $\delta = \frac{1}{2}$. Similarly for (3.36).

(iii) Bond-percolation on the triangular and the hexagonal lattice. In this application we take $\mathcal{G}$ = the triangular lattice and $\mathcal{G}_d$ = the hexagonal lattice, imbedded as in Ex. 2.6 (ii). Thus the vertices of $\mathcal{G}$ are at the points $(i_1, i_2\sqrt{3})$ and $(j_1 + \frac{1}{2}, (j_2 + \frac{1}{2})\sqrt{3}), i_1, i_2, j_1, j_2 \in \mathbb{Z}$. The faces of $\mathcal{G}$ are equilateral triangles and its edges are under an angle 0, $\pi/3$ or $2\pi/3$ with the first coordinate axis. The faces of $\mathcal{G}_d$ are regular hexagons and its edges are under angles $\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{3}, \frac{\pi}{2} + \frac{2\pi}{3}$ with the first coordinate axis. Strictly speaking, this is not a periodic imbedding, but as pointed out in Sect. 2.1 one merely has to change the vertical scale to make it periodic. In addition we shall describe this application in terms of bond-percolation. This is simpler than its equivalent formulation as a site-problem which can be obtained by going over to the covering graphs, as discussed in Sect. 2.5. Since $\mathcal{G}$ and $\mathcal{G}_d$ are a dual pair, their covering graphs form a matching pair. (See Sect. 2.6, especially Ex. 2.6 (ii).) One can verify this easily explicitly, but the covering graphs are more complicated than $\mathcal{G}$ and $\mathcal{G}_d$ themselves.

As we shall see below, for the one-parameter bond-problem on these graphs the critical probabilities are given by

$$(3.75) \qquad p_T(\mathcal{G};\text{bond}) = p_S(\mathcal{G};\text{bond}) = p_H(\mathcal{G};\text{bond}) = 2 \sin \frac{\pi}{18},$$

$$(3.76) \qquad p_T(\mathcal{G}_d;\text{bond}) = p_S(\mathcal{G}_d;\text{bond}) = p_H(\mathcal{G}_d;\text{bond}) = 1 - 2 \sin \frac{\pi}{18}.$$

Before we come to this result we describe first the 3-parameter problem of Sykes and Essam (1964). The edge set $\mathcal{E}$ of $\mathcal{G}$ is divided into the three classes

$$\mathcal{E}_i = \{\text{edges of } \mathcal{G} \text{ making an angle of } (i-1)\frac{\pi}{3}$$
$$\text{with first coordinate axis}\}, \; i = 1,2,3.$$

An edge of $\mathcal{E}_i$ is passable with probability $p(i)$. Each edge of $\mathcal{G}_d$ intersects exactly one edge of $\mathcal{G}$ and vice versa. In the covering graphs a pair of intersecting edges of $\mathcal{G}$ and $\mathcal{G}_d$ would correspond to one common vertex of the covering graphs. In accordance with this fact we take an edge of $\mathcal{G}_d$ as passable iff the edge of $\mathcal{G}$ which it intersects is passable. Thus, any configuration of passable and blocked edges in $\mathcal{G}$ is viewed at the same time as a configuration of passable and blocked edges on $\mathcal{G}_d$. The analogues of σ and σ^* in the bond version become

$$\sigma(\bar{n};i,p,\mathcal{G}) = P_p\{\exists \text{ crossing in the i-direction of } [0,n_1]\times[0,n_2] \text{ on } \mathcal{G} \text{ with all its edges passable}\},$$

$$\sigma^*(\bar{n};i,p,\mathcal{G}) = P_p\{\exists \text{ crossing in the i-direction of } [0,n_1]\times[0,n_2] \text{ on } \mathcal{G}^* \text{ with all its edges blocked}\}.$$

To verify condition A with this interpretation of σ and σ^* we follow Sykes and Essam's ingenious use of the star-triangle transformation. Instead of considering crossing probabilities on $\mathcal{G}_d$ itself, we consider crossing probabilities on a translate of $\mathcal{G}_d$, namely

$$\mathcal{H} := \mathcal{G}_d - (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}). \tag{3.77}$$

Of course we take the probability of an edge e of $\mathcal{H}$ being passable equal to the probability that the translated edge $e+(\frac{1}{2},\frac{1}{2\sqrt{3}})$ of $\mathcal{G}_d$ is passable. These probabilities are $p(1)$, $p(2)$ and $p(3)$ for the edges which make an angle of $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{\pi}{3}$ and $\frac{\pi}{2}+\frac{2\pi}{3}$ with the first coordinate axis, respectively. The vertex set of $\mathcal{H}$ coincides with that of $\mathcal{G}$ and each "up-triangle" of $\mathcal{G}$ (i.e., the closure of a triangular face F of $\mathcal{G}$ with vertices at $(i_1,i_2\sqrt{3})$, $(i_1+1,i_2\sqrt{3})$ and $(i_1+\frac{1}{2},(i_2+\frac{1}{2})\sqrt{3})$ for some $i_1,i_2 \in \mathbb{Z}$) contains a "star" of three edges of $\mathcal{H}$, one through each vertex on the perimeter of F (see Fig. 3.5).

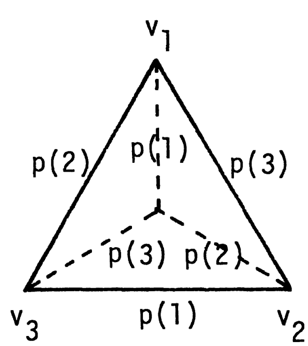

Figure 3.5 An up-triangle of $\mathcal{G}$ with a star of $\mathcal{H}$. —— = edges of $\mathcal{G}$, --- = edges of $\mathcal{H}$. The p-value next to an edge gives the probability for that edge to be passable.

It turns out that the connectivity properties on $\mathcal{G}$ and $\mathcal{H}$ can be made identical by a suitable matching of certain probabilities in each up-triangle separately. Note that it is not so much the full configuration of passable edges in each up-triangle that is important, as the pairs of vertices which are connected in each up-triangle. Here we make the convention that two vertices v_1 and v_2 on the perimeter of F are connected in $\overline{F}$ on $\mathcal{G}(\mathcal{H})$ if one can go along passable edges of $\mathcal{G}$ (blocked edges of $\mathcal{H}$) in $\overline{F}$ from v_1 to v_2. If one has a sequence $v_0,\dots,v_\nu$ of distinct vertices of $\mathcal{G}$ (or $\mathcal{H}$) such that v_{j-1} and v_j are connected in the unique up-triangle to which they both belong, $j = 1,\dots,\nu$, then there exists a passable path $(w_0,e_1,\dots,e_\tau,w_\tau)$ on $\mathcal{G}$ (or a path on $\mathcal{H}$ with all its edges blocked) with endpoints $w_0 = v_0$, $w_\tau = v_\nu$ and which contains the vertices $v_0,\dots,v_\nu$ but only enters up-triangles which have one of the v_i as vertices. Since the diameter of any up-triangle equals one, this together with (3.77) implies

$$(3.78)\quad \sigma^*(\overline{n}-\overline{\rho};1,p,\mathcal{G}) = P_p\{\exists \text{ a horizontal crossing of } [-\tfrac{1}{2},n_1-\rho_1-\tfrac{1}{2}]\times[-\tfrac{1}{2\sqrt{3}},n_2-\rho_2-\tfrac{1}{2\sqrt{3}}] \text{ on } \mathcal{H} \text{ all of whose edges are blocked}\}$$

$$\geq P_p\{\exists \text{ a sequence of vertices } v_0,\dots,v_\nu \text{ such that } v_{j-1} \text{ and } v_j \text{ are connected on } \mathcal{H},\ 1\leq j\leq\nu, \text{ and}$$

$$v_s \in [-\tfrac{3}{2},n_1-\rho_1+\tfrac{1}{2}]\times[-\tfrac{1}{2\sqrt{3}}+1,n_2-\rho_2-\tfrac{1}{2\sqrt{3}}-1],\ 1\leq s\leq\nu-1,$$

$$\text{while } v_0(1) < -\tfrac{3}{2},\ v_\nu(1) > n_1-\rho_1+\tfrac{1}{2}\}\ .$$

If the event in the last member of (3.78) occurs and $r = (w_0,e_1,\dots,e_\tau,w_\tau)$ is the path on $\mathcal{H}$ through $v_0,\dots,v_\nu$ as above, then r contains a horizontal crossing of

$$[-\tfrac{1}{2},n_1-\rho_1-\tfrac{1}{2}]\times[-\tfrac{1}{2\sqrt{3}},n_2-\rho_2-\tfrac{1}{2\sqrt{3}}]$$

with all edges blocked. Assume now that p_0 is such that for an up-triangle F with vertices v_1,v_2,v_3 and any subset Γ of $\{1,2,3\}$

(3.79) P_{p_0} {the pairs of vertices connected in $\bar{F}$ on $\mathcal{G}$ are exactly the pairs v_i, v_j with $i,j \in \Gamma$}

$= P_{p_0}$ {the pairs of vertices connected in $\bar{F}$ on $\mathcal{H}$ are exactly the pairs v_i, v_j with $i,j \in \Gamma$} .

Then the right hand side of (3.78) remains unchanged for $p = p_0$ if $\mathcal{H}$ is replaced by $\mathcal{G}$, because distinct up-triangles have no edges in common, and have consequently independent edge configurations. (This holds on $\mathcal{H}$ as well as on $\mathcal{G}$.) However, when $\mathcal{H}$ is replaced by $\mathcal{G}$ the last member of (3.78) is at least equal to

$$P_p\{\exists \text{ a passable horizontal crossing of } [-\tfrac{1}{2}, n_1-\rho_1-\tfrac{1}{2}] \times [-\tfrac{1}{2\sqrt{3}}\, n_2-\rho_2-\tfrac{1}{2\sqrt{3}}] \text{ on } \mathcal{G}\}$$

$$\geq \sigma((n_1-\rho_1+1, n_2-1-\rho_2), 1, p).$$

Therefore (3.35) holds when $\rho_1 \leq 1$, $\rho_2 \geq -1$ for any p_0 which satisfies (3.79). Similarly for (3.36), and consequently Condition A is implied by (3.79).

We shall now verify that (3.79) holds for all $p_0 \in \mathcal{S}$, where

$$\mathcal{S} = \{p \in \mathcal{P}_3 : 0 << p << 1, p(1)+p(2)+p(3)-p(1)p(2)p(3) = 1\}. \tag{3.80}$$

The only possibilities for Γ are ϕ, {1,2,3} and the three subsets of {1,2,3} consisting of exactly one pair. These last three subsets and their probabilities can be obtained from each other by cyclical permutations of the indices, so that it suffices to consider $\Gamma = \phi$, $\Gamma = \{1,2\}$ and $\Gamma = \{1,2,3\}$. For $\Gamma = \phi$, the left and right hand side of (3.79) are, respectively,

$$(1-p(1))(1-p(2))(1-p(3)) \tag{3.81}$$

and

$$p(1)p(2)p(3) + p(1)p(2)(1-p(3)) + p(1)(1-p(2))p(3) + (1-p(1))p(2)p(3) \tag{3.82}$$

(recall that on $\mathcal{H}$ we are looking for paths with <u>blocked</u> edges). It is simple algebra to check that (3.81) and (3.82) are equal for $p \in \mathcal{S}$. Equation (3.79) for $\Gamma = \{1,2,3\}$ again reduces to the equality of

(3.81) and (3.82). Finally, if $\Gamma = \{1,2\}$ and the vertices are numbered as in Fig. 3.5, then both sides of (3.79) equal

$$p(3)(1-p(1))(1-p(2)).$$

The above shows that in this example Condition A holds whenever $p_0 \in \mathcal{S}$. Unfortunately, neither of the coordinate axes is an axis of symmetry for the sets $\mathcal{E}_2$ and $\mathcal{E}_3$ and therefore Theorem 1 cannot be used for this 3-parameter problem. To obtain the required amount of symmetry we have to restrict ourselves to the two-parameter problem with $p(2) = p(3)$. In this case Theorem 1 applies, and for this problem the critical surface in $\mathcal{P}_2$ is obtained by taking $p(2) = p(3)$ in (3.80). Thus, if we take

$$P_p\{e \text{ is passable}\} = \begin{cases} p(1) & \text{if } e \in \mathcal{E}_1 \\ p(2) & \text{if } e \in \mathcal{E}_2 \cup \mathcal{E}_3 , \end{cases}$$

then there are infinite passable clusters on the trianglar lattice $\mathcal{G}$ under P_p with $0 < p(1), p(2) < 1$ iff

(3.83) $$p(1) + 2p(2) - p(1)p(2)^2 > 1.$$

When restricted further to the one-parameter problem with $p(1) = p(2) = p(3)$ we find for the triangular lattice the critical probabilities given in (3.75) since $2 \sin \frac{\pi}{18}$ is the unique root in $(0,1)$ of $3p-p^3 = 1$. This value was conjectured by Sykes and Essam (1964) and first rigorously confirmed by Wierman (1981). By interchanging the role of "passable" and "blocked" one finds for the one-parameter problem on the hexagonal lattice the critical values given in (3.76). Of course, by obvious isomorphisms these results determine the percolative region also when we take $p(1) = p(2)$ or $p(1) = p(3)$ instead of $p(2) = p(3)$.

So far we have been unable to prove the full conjecture of Sykes and Essam (1964) that $\mathcal{S}$ is the critical surface for the three-parameter problem. There are, however, many indications that the conjecture is correct, in addition to the above verification for the two-parameter problem. First, one can prove that no percolation can occur on $\mathcal{G}$ if

$$p(1) + p(2) + p(3) - p(1)p(2)p(3) \leq 1.$$

Thus, the percolative region is contained in $\mathcal{P}_+$ (see (3.58) for notation), and its intersection with the plane $\{p(2) = p(3)\}$ is the same as the intersection of $\mathcal{P}_+$ with this plane. Also, if we take $p(3) = 0$, then the bond-problem on $\mathcal{G}$ reduces to the bond-problem on $\mathbb{Z}^2$ with probabilities $p(1)$ and $p(2)$ for horizontal and vertical edges to be passable. This is evident if we imbed the triangular lattice as in Fig. 2.5 in Ex. 2.1 (iii). However, by Application (ii) above we know that the critical surface for this bond-problem on $\mathbb{Z}^2$ is given by (3.71), which is precisely the restriction of (3.80) to $p(3) = 0$, (if we ignore the requirement $p(3) > 0$). Last, we can modify the three parameter problem slightly so that the first coordinate axis becomes an axis of symmetry. To do this we interchange the role of $p(2)$ and $p(3)$ in every second row of up-triangles. To be precise we leave $\mathcal{E}_1$ as before but replace $\mathcal{E}_2$ and $\mathcal{E}_3$ by

(3.84) $\mathcal{E}_2' = \{e$: e an edge between $(i_1, i_2\sqrt{3})$ and $(i_1 + \frac{1}{2}, (i_2 + \frac{1}{2})\sqrt{3})$ or between $(i_1, i_2\sqrt{3})$ and $(i_1 + \frac{1}{2}, (i_2 - \frac{1}{2})\sqrt{3})$ for some $i_1, i_2 \in \mathbb{Z}\}$

(3.85) $\mathcal{E}_3' = \{e$: e an edge between $(i_1, i_2\sqrt{3})$ and $(i_1 - \frac{1}{2}, (i_2 + \frac{1}{2})\sqrt{3})$ or between $(i_1, i_2\sqrt{3})$ and $(i_1 - \frac{1}{2}, (i_2 - \frac{1}{2})\sqrt{3})$ for some $i_1, i_2 \in \mathbb{Z}\}$

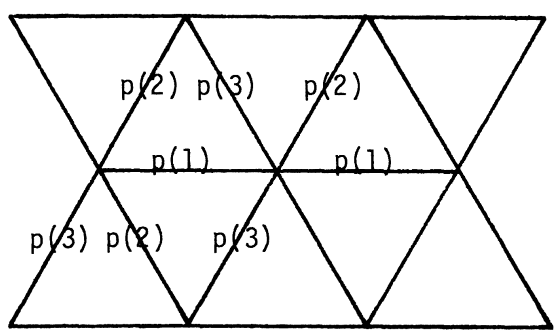

Figure 3.6 A modified 3-parameter bond-problem on the triangular lattice. The p-value next to an edge gives the probability for that edge to be passable.

The assignment of probabilities becomes as indicated in Fig. 3.6. We can define $\mathcal{H}$ as before, but the probabilities of an edge of $\mathcal{H}$ being passable have to be modified in accordance with (3.84) and (3.85). Each up-triangle will have an assignment of probabilities as in Fig. 3.5 or with p(2) and p(3) interchanged. However, the remainder of the argument showing that Condition A holds whenever $p_0 \in \mathcal{S}$ remains unchanged. Since this new example has the first-coordinate axis as axis of symmetry for $\mathcal{G}$, as well as for the edge classes $\mathcal{E}_1, \mathcal{E}_2', \mathcal{E}_3'$, Theorem 1 and Cor. 1 applies. Thus, $\mathcal{S}$ is the critical surface for the modified 3-parameter problem.

(iv) Site-percolation on $\mathbb{Z}^2$. In this example we shall verify Condition B. It will, however, not lead to an explicit determination of the percolative region. For our graph $\mathcal{G}$ we take the quadratic lattice $\mathcal{G}_0$ of Ex. 2.1 (i). We consider the two-parameter site-percolation problem corresponding to

$$\mathcal{V}_1 = \{(i_1,i_2): i_1+i_2 \text{ is even}\} \quad ,$$

$$\mathcal{V}_2 = \{(i_1,i_2): i_1+i_2 \text{ is odd}\} \quad .$$

A trivial change of scale by a factor $\frac{1}{2}$ in both the horizontal and vertical direction is required to bring this problem in the periodic form (3.18), but this will not change the fact that $\mathcal{G}, \mathcal{G}^*, \mathcal{V}_1$ and $\mathcal{V}_2$ are unchanged by reflection in a coordinate axis or in the 45° line $x(1) = x(2)$ (see Fig. 3.1). Thus, both coordinate axes are axes of symmetry while (3.52)-(3.55) hold trivially when all a_j-d_j are equal to one and $h(x) = x$, because the probability of an occupied horizontal crossing of $[0,n]\times[0,m]$ on $\mathcal{G}$ is the same as the probability of an occupied vertical crossing of $[0,m]\times[0,n]$. Similarly for vacant crossings on $\mathcal{G}^*$. Thus, Theorem 3.2 and Cor. 3.1 apply, and the critical surface $\mathcal{S}$ is given in this example by

$$(3.86) \qquad \mathcal{S} = \{p_0 = (p_0(1),p_0(2)): 0 \ll p_0 \ll 1,\ p_0 = t_0(p_1)p_1$$
$$\text{for } p_1 \text{ of the form } (1,p) \text{ or } (p,1), \text{ with } 0 < p \le 1\} \ ,$$

where

$$(3.87) \qquad t_0(p_1) = \inf\{t \ge 0: tp_1 \in \mathcal{P}_2,\ \limsup \sigma((n,n);1,tp_1) > 0\} \ .$$

Infinite occupied clusters on $\mathcal{G}$ can occur only for $p \in \mathcal{P}_+$ (see (3.58) with $\lambda = 2$), while for $p \in \mathcal{P}_-$

(3.88) $$E_p\{\#W(v)\} < \infty .$$

When restricted to the one-parameter problem $p(1) = p(2)$ Theorem 3.2 (together with (3.66)) implies (see Ex. 2.2 (i) for $\mathcal{G}_0^*$)

(3.89) $$p_T(\mathcal{G}_0) = p_S(\mathcal{G}_0) = p_H(\mathcal{G}_0) = 1-p_T(\mathcal{G}_0^*) = 1-p_S(\mathcal{G}_0^*) = 1-p_H(\mathcal{G}_0^*).$$

This result was recently proved by Russo (1981).

It is also interesting to see how $\mathcal{S}$ behaves near the edges $p(1) = 1$ and $p(2) = 1$ of $\mathcal{P}_2$. For $p(1) = 1$, the occupancy of a path is determined only by the vertices from $\mathcal{V}_2$ on the path. From this it follows that the questions whether $\theta(p,v) > 0$ or $E_p\{\#W(v)\} < \infty$ reduce to the same questions in a one-parameter problem with $p = p(2)$ on the graph $\mathcal{H}$ with vertex set $\mathcal{V}_2$ and with $(i_1,i_2) \in \mathcal{V}_2$ adjacent to $(j_1,j_2) \in \mathcal{V}_2$ on $\mathcal{H}$ iff

$$|i_1-j_1| = 1 \text{ and } |i_2-j_2| = 1$$

or

$$i_1 = j_1, \; |i_2-j_2| = 2$$

or

$$|i_1-j_1| = 2, \; i_2 = j_2 .$$

This graph is drawn in Fig. 3.7, together with $\mathcal{G}_0$.

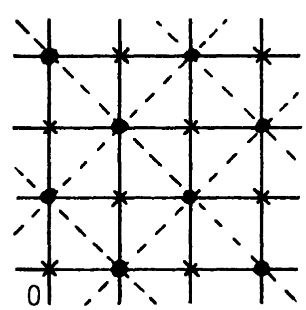

Figure 3.7 $\mathcal{H}$ has vertices at the circles only; its edges are the solid as well as the dashed segments; $\mathcal{G}_0$ has vertices at the circles and at the stars; its edges are the solid segments only.

Clearly $\mathcal{H}$ is isomorphic to $\mathcal{G}_0^*$ (see Ex. 2.2 (i)) and therefore on $p(1) = 1$, infinite occupied clusters occur if and only if $p(2) > p_H(\mathcal{G}_0^*)$; moreover $E_p\{\#W(v)\} < \infty$ for $p(2) < p_H(\mathcal{G}_0^*)$. Simple Peierls arguments (i.e., counting arguments such as in Broadbent and Hammersley (1957), Theorem 7 and Hammersley (1959), Theorem 1) establish that

$$0 < p_H(\mathcal{G}_0^*) < 1 .$$

Thus, for any $0 < p(2) < p_H(\mathcal{G}_0^*)$ and $p = (1,p(2))$ (3.88) holds. Moreover, as we shall see in the proof of Lemma 5.4, $p >> 0$ and (3.88) imply that $\sigma((n,n);i,p) \to 0$. Since (3.88) for any p implies that (3.88) is also valid for any p' with $p'(i) \leq p(i)$, $i = 1,2$ (see Lemma 4.1), it follows that $\mathcal{S}$ cannot have any accumulation points in $\{1\} \times [0,p_H(\mathcal{G}_0^*))$. Interchanging the role of $p(1)$ and $p(2)$ we see that $\mathcal{S}$ has no accumulation points in $[0,p_H(\mathcal{G}_0^*)) \times \{1\}$ either. Furthermore, it will be shown in Ch. 10 (see Ex. 10.2 (i)) that in the interior of $\mathcal{P}_2$ $\mathcal{S}$ lies strictly above the line $p(1) + p(2) = 1$. Thus, $\mathcal{S}$, $\mathcal{P}_+$ and $\mathcal{P}_-$ should look more or less as indicated in Fig. 3.8.

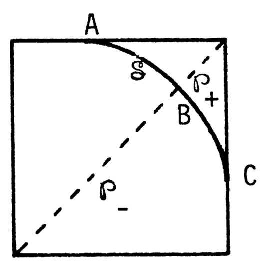

Figure 3.8 $A = (P_H(\mathcal{G}_0^*),1)$, $B = (P_H(\mathcal{G}_0),P_H(\mathcal{G}_0))$,

$C = (1,P_H(\mathcal{G}_0^*))$

The points $A = (p_H(\mathcal{G}_0^*),1)$ and $C = (1,p_H(\mathcal{G}_0^*))$ are the points on the boundary of $\mathcal{P}_2$ in the closure of $\mathcal{S}$, while $\mathcal{S}$ intersects the diagonal $p(1) = p(2)$ in $B = (p_H(\mathcal{G}_0),\ p_H(\mathcal{G}_0)) = (1-p_H(\mathcal{G}_0^*),1-p_H(\mathcal{G}_0^*))$.

(v) For a last application we consider one-parameter site-percolation on the diced lattice of Ex. 2.1 (v). We shall show that this graph satisfies (3.52)-(3.55) so that Theorem 3.2 applies. For p_1 we can take any number in $(0,1)$. We then find from (3.43)-(3.51) and the definition (3.56)

$$p_0 = \inf\{p \geq 0: \limsup_{n\to\infty} \sigma((n,a_1 n);1,p) > 0 \text{ or } \limsup_{n\to\infty} \sigma((n,b_1 n);2,p) > 0\}$$

$$= p_H \text{ (diced lattice)} = p_T \text{ (diced lattice)}$$

$$= 1-p_H \text{ (matching graph of diced lattice)}$$

$$= 1-p_T \text{ (matching graph of diced lattice)} .$$

Note that the diced lattice is itself a mosaic, $\mathcal{D}$ say. Therefore, the diced lattice $\mathcal{D}$ is the first graph of the matching pair $(\mathcal{D},\mathcal{D}^*)$ based on $(\mathcal{D},\emptyset)$ (Comment 2.2 (vi)). In the imbedding of Ex. 2.1 (v) the diced lattice is clearly invariant under a rotation over 120°, and this will also be true for $\mathcal{D}^*$, where $\mathcal{D}^*$ is obtained by inserting the "diagonal edges" in each face of $\mathcal{D}$. From this property it is easy to derive (3.52)-(3.55) with $h(x) = x$. We content ourselves with demonstrating (3.52). Note that any horizontal crossing on $\mathcal{D}$ of $B = [0,n] \times [0,\frac{1}{4}n]$ contains a continuous curve ψ inside B and connecting the left and right edge of B. When B is rotated around the origin over 120° it goes over into the rectangle $\tilde{B}$ with vertices 0, $P_1 := (-\frac{n}{2},\frac{n}{2}\sqrt{3})$, $P_2 := (-\frac{n}{2}-\frac{n}{8}\sqrt{3}, \frac{n}{2}\sqrt{3}-\frac{n}{8})$, $P_3 := (-\frac{n}{8}\sqrt{3}, -\frac{n}{8})$. $\tilde{\psi}$ goes

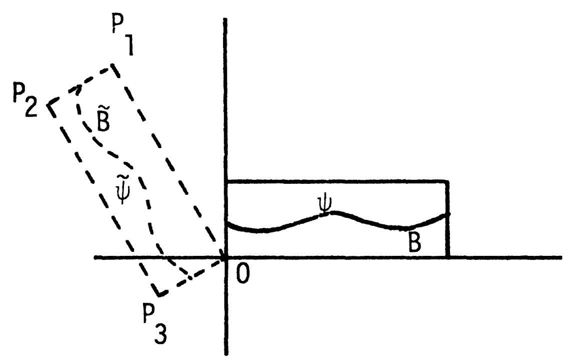

Figure 3.9

over into a continuous curve on $\mathcal{D}$ inside $\tilde{B}$ and connecting the segment from 0 to P_3 with the segment from P_1 to P_2. In particular $\tilde{\psi}$ begins below the first coordinate-axis $(x(2) = 0)$ and ends above the horizontal line through P_2, $x(2) = \frac{n}{2}\sqrt{3} - \frac{n}{8}$. Also ψ is contained between the vertical lines through P_2, $x(1) = -\frac{n}{2} - \frac{n}{8}\sqrt{3}$, and the vertical line through 0, $x(1) = 0$. In particular, $\tilde{\psi}$ contains a vertical crossing of

$$[-\tfrac{n}{2} - \tfrac{n}{8}\sqrt{3} - 1, 1] \times [\Lambda, \tfrac{n}{2}\sqrt{3} - \tfrac{n}{8} - \Lambda] \quad ,$$

if $\Lambda \geq$ length of any edge of $\mathcal{G}$. By the invariance of $\mathcal{G}$ and P_p under the rotation over 120° we therefore have

$$\text{(3.90)} \qquad P_p\{\exists \text{ occupied vertical crossing of } [-\tfrac{n}{2} - \tfrac{n}{8}\sqrt{3} - 1, 1] \times [\Lambda, \tfrac{n}{2}\sqrt{3} - \Lambda]\}$$

$$\geq P_p\{\exists \text{ continuous curve } \tilde{\psi} \text{ in } \tilde{B} \text{ on } \mathcal{G} \text{ connecting the segment from } 0 \text{ to } P_3 \text{ with the segment from } P_1 \text{ to } P_2 \text{ and with all vertices on } \psi \text{ occupied}\}$$

$$\geq P_p\{\exists \text{ occupied horizontal crossing of } [0,n] \times [0, \tfrac{n}{4}]\}.$$

This is essentially (3.52), since by the periodicity of $\mathcal{G}$ with periods $(\sqrt{3}, 0)$, $(0,3)$ the left hand side of (3.90) is at most

$$\text{(3.91)} \qquad P_p\{\exists \text{ occupied vertical crossing of } [0, n(\tfrac{1}{2} + \tfrac{\sqrt{3}}{8}) + 2 + \sqrt{3}] \times [0, \tfrac{n}{2}\sqrt{3} - \tfrac{n}{8} - 2\Lambda - 3]\}$$

$$\leq P_p\{\exists \text{ occupied vertical crossing of } [0,n] \times [0, \tfrac{n}{2}]\}$$

for large n (use Comment 3.3 (v)). For the imbedding of $\mathcal{G}$ of Ex.2.1 (v) this would say

$$\text{(3.92)} \qquad \sigma((n, \tfrac{n}{2}); 2, p, \mathcal{G}) \geq \sigma((n, \tfrac{n}{4}); 1, p, \mathcal{G}).$$

This is actually not the inequality which we can use, because we first have to change scale in order to make $\mathcal{G}$ periodic with periods $(1,0)$ and $(0,1)$. This, however, does not change the form of the inequality (3.92), and hence (3.52) follows for some a_1-a_3 and $h(x) = x$.