2. WHICH GRAPHS DO WE CONSIDER?

This chapter discusses the graphs with which we shall work, as well as several graph-theoretical tools. Except for the basic definitions in Sect. 2.1-2.3 the reader should skip the remaining parts of this chapter until the need for them arises.

2.1 Periodic graphs.

Throughout this monograph we consider only graphs which are imbedded in $\mathbb{R}^d$ for some $d < \infty$. Only when strictly necessary shall we make a distinction between a graph and its image under the imbedding. Usually we denote the graph by $\mathcal{G}$, a generic vertex of $\mathcal{G}$ by u, v or w (with or without subscripts), and a generic edge of $\mathcal{G}$ by e, f or g (with or without subscripts). "Site" will be synonymous with "vertex", and "bond" will be synonymous with "edge". The collection of vertices of $\mathcal{G}$ will always be a countable subset of $\mathbb{R}^d$. The collection of edges of $\mathcal{G}$ will also be countable, and each edge will be a simple arc - that is, a homeomorphic image of the interval $[0,1]$ - in $\mathbb{R}^d$, with two vertices as endpoints but no vertices of $\mathcal{G}$ in its interior. In particular we take an edge to be closed, i.e., we include the endpoints in the edge. If e is an edge, then we denote its interior, i.e., e minus its endpoints, by $\mathring{e}$. We shall say that e is <u>incident to</u> v if v is an endpoint of e. We only allow graphs in which the endpoints of each edge are distinct; thus we assume

(2.1) $\qquad\qquad \mathcal{G}$ contains no loops.

We shall, however, allow several edges between the same pair of distinct vertices.

The notation

$$v_1 \mathcal{G} v_2 \quad \text{or equivalently} \quad v_2 \mathcal{G} v_1$$

will be used to denote that v_1 and v_2 are <u>adjacent</u> or <u>neighbors</u> on $\mathcal{G}$. This means that there exists an edge of $\mathcal{G}$ with endpoints v_1 and v_2.

A <u>path</u> on $\mathcal{G}$ will be a sequence $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ with $v_0, \dots, v_\nu$ vertices of $\mathcal{G}$ and $e_1, \dots, e_\nu$ edges of $\mathcal{G}$ such that e_{i+1} is an edge with endpoints v_i and v_{i+1}. We call v_0 (v_ν) the first or initial (last or final) vertex of r and say that r is a path from v_0 to v_ν. The path r is called <u>self-avoiding</u> if all its vertices are distinct. <u>Unless otherwise stated all paths are tacitly taken to be self-avoiding</u>. In the few cases where we have to deal with paths which are not necessarily self-avoiding we shall call them <u>paths with possible double points</u>. If $\mathcal{G}$ is any graph then we can always turn a path with possible double points $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ into a self-avoiding sub-path $\tilde{r}$ with the same initial and final vertex as r. This is done by the process of <u>loop-removal</u> which works as follows: Let ρ_1 be the smallest index for which there exists a $\tau_1 > \rho_1$ with $v_{\tau_1} = v_{\rho_1}$. From the possible τ_1 with this property choose the largest one. Form the path $r_1 = (v_0, e_1, \dots, v_{\rho_1} = v_{\tau_1}, e_{\tau_1+1}, v_{\tau_1+1}, \dots, e_\nu, v_\nu)$ by removal of the "loop" $(v_{\rho_1}, e_{\rho_1+1}, v_{\rho_1+1}, \dots, e_{\tau_1})$. The piece $(v_0, \dots, v_{\rho_1})$ of r is free of double-points and,by the maximality of τ_1, it is not hit again by the remaining piece $(e_{\tau_1+1}, \dots, e_\nu, v_\nu)$ of r_1. Thus, if r_1 still has a double point there have to exist $\tau_1 < \rho_2 < \tau_2$ with $v_{\rho_2} = v_{\tau_2}$. Again we choose the smallest such ρ_2 and then the largest τ_2 for that ρ_2, and remove from r_1 the piece $(v_{\rho_2}, e_{\rho_2+1}, \dots, e_{\tau_2})$ to obtain another subpath r_2 of r_1. We continue in this way until we arrive at a path $\tilde{r}$ without double points. One easily verifies that removal of a loop neither changes the first nor the last vertex of a path.

In addition to (2.1) we shall almost always impose the conditions (2.2)-(2.5) below on our graphs:

(2.2) $\mathcal{G}$ is imbedded in $\mathbb{R}^d$ in such a way that each coordinate vector of $\mathbb{R}^d$ is a period for the image.

By (2.2) we mean that $v \in \mathbb{R}^d$ is a vertex of (the image of) $\mathcal{G}$ iff $v + \sum_1^d k_i \xi_i$ is a vertex of $\mathcal{G}$ for all $k_i \in \mathbb{Z}$, where ξ_i denotes the ith coordinate vector of $\mathbb{R}^d$. Also, $e \subset \mathbb{R}^d$ is an edge of (the image of) $\mathcal{G}$ iff $e + \sum_1^d k_i \xi_i$ is an edge for all $k_i \in \mathbb{Z}$.

(2.3) There exists a $z < \infty$ such that there are at most z edges of $\mathcal{G}$ incident to any vertex of $\mathcal{G}$.

(2.4) All edges of $\mathcal{G}$ have finite diameter. Every compact set of $\mathbb{R}^d$ intersects only finitely many edges of $\mathcal{G}$.

(2.5) $\mathcal{G}$ is connected.

Of course (2.5) means that for every pair of vertices v_1, v_2 of $\mathcal{G}$ there exists a path on $\mathcal{G}$ from v_1 to v_2.

Def. 1. A periodic graph $\mathcal{G}$ is a graph which is imbedded in some $\mathbb{R}^d$, $d < \infty$, such that (2.1)-(2.5) hold. ///

The name "periodic graph" is a bit of a misnomer. It is really the imbedding which is periodic. It will be obvious from Ch. 3 that our percolation problems depend only on the abstract structure of the graph $\mathcal{G}$, and not on its imbedding. For the proofs it is often advantageous to change the imbedding from a standard one, by mapping $\mathbb{R}^d$ onto itself by an affine isomorphism. As stated, this does not effect the percolation theory problems. We illustrate with some standard examples.

Examples.

(i) One of the most familiar graphs is the simple quadratic lattice. It is imbedded in $\mathbb{R}^2$; its vertex set is $\mathbb{Z}^2$, and the edges are the straight-line segments between (i_1, i_2) and $(i_1 \pm 1, i_2)$ and between (i_1, i_2) and $(i_1, i_2 \pm 1)$, $i_1, i_2 \in \mathbb{Z}$. Thus, two vertices (i_1, i_2) and (j_1, j_2) $(i_r, j_r \in \mathbb{Z})$ are neighbors iff

$$|i_1 - j_1| + |i_2 - j_2| = 1. \tag{2.6}$$

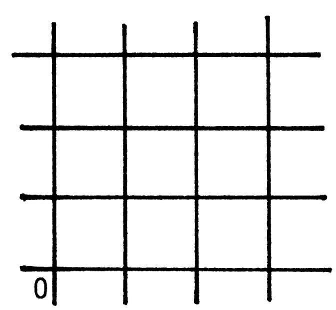


Figure 2.1 $\mathcal{G}_0$

This is a periodic graph, and we denote it by $\mathcal{G}_0$ throughout this monograph.

(ii) For bond-percolation on $\mathbb{Z}^2$ one wants to use the graph which is obtained by adding diagonals to alternating squares in $\mathcal{G}_0$.

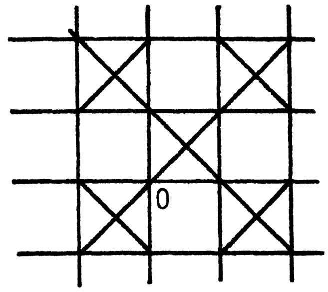


Figure 2.2

The formal description for this imbedding in $\mathbb{R}^2$ is that the vertex set is $\mathbb{Z}^2$, and two vertices $v = (i_1,i_2)$ and $w = (j_1,j_2)$ $(i_r,j_r \in \mathbb{Z})$ are neighbors iff (2.6), or (2.7) below holds.

(2.7) $\quad |i_1-j_1| = |i_2-j_2| = 1$ and i_1+i_2 is odd or i_1-i_2 is even.

Under this imbedding we do not have a periodic graph, because the periods are only $2\xi_1$ and $2\xi_2$, where $\xi_1 = (1,0)$, $\xi_2 = (0,1)$ are the coordinate vectors of $\mathbb{R}^2$. For our purposes the preferred imbedding, which does give us a periodic graph, is obtained by translating the coordinate system in Fig. 2.2 by the vector $(\frac{1}{2},\frac{1}{2})$, rotating it over 45° and changing the scale by a factor $\sqrt{2}$. This gives us the periodic graph which we shall call $\mathcal{G}_1$, and which is drawn in Fig. 2.3.

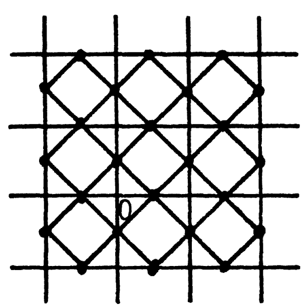


Figure 2.3 $\mathcal{G}_1$

The vertices of $\mathcal{G}_1$ are located at the points $(i_1+\frac{1}{2},i_2)$ and $(i_1,i_2+\frac{1}{2})$, $i_1,i_2 \in \mathbb{Z}$. Two vertices $v = (v(1),v(2))$ and $w = (w(1),w(2))$ are adjacent iff

(2.8) $\quad v(1) = w(1) \in \mathbb{Z}, v(2),w(2) \in \mathbb{Z}+\frac{1}{2}$ and $|v(2)-w(2)| = 1$

or

(2.9) $\quad v(2) = w(2) \in \mathbb{Z}, v(1),w(1) \in \mathbb{Z}+\frac{1}{2}$ and $|v(1)-w(1)| = 1$

or

(2.10) $\quad |v(1)-w(1)| = |v(2)-w(2)| = \frac{1}{2}$.

(iii) Another familiar example is the socalled triangular lattice: Divide $\mathbb{R}^2$ into equilateral triangles by means of the horizontal lines $x(2) = \frac{k}{2}\sqrt{3}$, $k \in \mathbb{Z}$, and lines under an angle of 60° or 120° with the first coordinate-axis through the points $(k,0)$, $k \in \mathbb{Z}$, see Fig. 2.4.

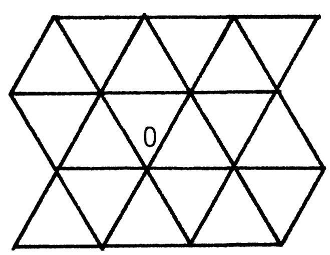


Figure 2.4

The vertices of the graph are the vertices of the equilateral triangles, and two such vertices are adjacent iff they are vertices of one and the same triangle. Even though the vector $(1,0)$ is a period for this imbedding, the vector $(0,1)$ is not. However, if we change the vertical scale by a factor $1/\sqrt{3}$ we obtain a periodic graph in $\mathbb{R}^2$. We denote it by $\mathcal{J}$. Its vertices are located at all points of the form (i_1,i_2) or $(i_1+\frac{1}{2},i_2+\frac{1}{2})$, $i_1,i_2 \in \mathbb{Z}$; each vertex has six neighbors. The six neighbors of the vertex v are

(2.11) $\quad v+(1,0),\ v+(\frac{1}{2},\frac{1}{2}),\ v+(-\frac{1}{2},\frac{1}{2}),$

$\quad v+(-1,0),\ v+(-\frac{1}{2},-\frac{1}{2}), v+(\frac{1}{2},-\frac{1}{2}).$

An amusing imbedding for the same graph - which we shall not use - is illustrated in Fig. 2.5. It shows that we can view $\mathcal{G}_0$ of ex. (i) as a subgraph of $\mathfrak{T}$

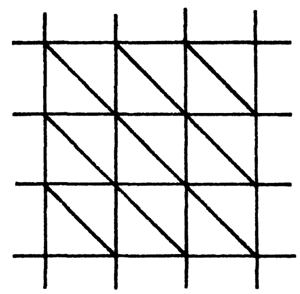


Figure 2.5

(iv) Another graph which we shall use for occasional illustrations is the hexagonal or honeycomb lattice. The usual way to imbed the hexagonal lattice is such that its faces are regular hexagons as illustrated in Fig. 2.6. The vertices are at the points

$$((k_1 + \frac{\ell}{2})\sqrt{3} + \cos(\frac{\pi}{6} + \frac{2\pi j}{6}), 3(k_2 + \frac{\ell}{2}) + \sin(\frac{\pi}{6} + \frac{2\pi j}{6})),$$

$$k_1, k_2 \in \mathbb{Z}, \ \ell = 0,1, \ 0 \le j \le 5.$$

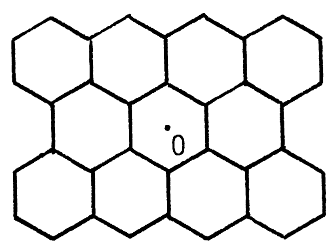


Figure 2.6 Hexagonal or honeycomb lattice

The origin is in the center of one of the hexagons and the periods are $(\sqrt{3},0)$ and $(0,3)$. We shall usually refer to this imbedding of the hexagonal lattice, even though it does not satisfy (2.2). We leave it to the reader to change scale so that (2.2) does become true.

(v) Our final example is the diced lattice, which is somewhat less familiar. We obtain it from the hexagonal lattice with vertices as in the last example by adding for each k_1, k_2, ℓ a vertex at $((k_1 + \frac{\ell}{2})\sqrt{3}, 3(k_2 + \frac{1}{2}\ell))$ (the center of one of the regular hexagons in Fig. 2.6) and connecting it to the three vertices

$$((k_1 + \frac{\ell}{2})\sqrt{3} + \cos(\frac{\pi}{6} + \frac{2\pi j}{6}), 3(k_2 + \frac{\ell}{2}) + \sin(\frac{\pi}{6} + \frac{2\pi j}{6})),\ j = 1,3,5.$$

This is illustrated in Fig. 2.7. The periods of this imbedding are again $(\sqrt{3},0)$ and $(0,3)$.

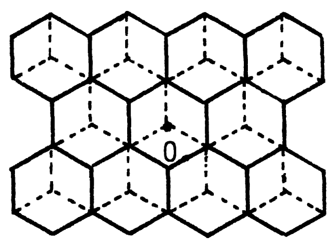


Figure 2.7 The diced lattice. It is obtained from the hexagonal lattice (solid lines) by adding the dashed edges.

2.2 Matching pairs.

With the exception of Ch. 5 we shall restrict ourselves to a special class of graphs imbedded in the plane. This class was introduced by Sykes and Essam (1964) and will be described in this section.

Def. 2. A mosaic $\mathcal{M}$ is a graph imbedded in $\mathbb{R}^2$ such that (2.1) and (2.4) hold, such that any two edges of $\mathcal{M}$ are either disjoint or have only one or two endpoints in common (these common endpoints are necessarily vertices of $\mathcal{M}$), and such that each component of $\mathbb{R}^2 \setminus \mathcal{M}$ is bounded by a Jordan curve made up of a finite number of edges of $\mathcal{M}$.

Comments.

(i) A graph which can be imbedded in $\mathbb{R}^2$ such that any two edges can have only endpoints in common is called planar. Thus, any mosaic is a planar graph. If a planar graph $\mathcal{M}$ is imbedded in such a way, then one calls each component of $\mathbb{R}^2 \setminus \mathcal{M}$ a face of $\mathcal{M}$.

(ii) The precise meaning of a "curve made up of edges of $\mathcal{M}$" is as follows: Let $e_1, \ldots, e_\nu$ be edges of $\mathcal{M}$ given by the homeomorphisms $\phi_i : [0,1] \to \mathbb{R}^d$, and such that

$$\phi_i(1) = \phi_{i+1}(0), \tag{2.12}$$

i.e., the final point of e_i equals the initial point of e_{i+1}. A curve J made up from $e_1, \ldots, e_\nu$, or obtained by (successively) traversing a piece of $e_1, e_2, \ldots, e_{\nu-1}$, and a piece of e_ν is a curve which can be represented by a map $\psi : [0,1] \to \mathbb{R}^d$ of the following form: for some $0 \le a < 1$, $0 < b \le 1$:

$$\psi(t) = \begin{cases} \phi_1(a + \nu(1-a)t) , & 0 \le t \le \frac{1}{\nu} , \\ \phi_{i+1}\left(\nu\left(t - \frac{i}{\nu}\right)\right) , & \frac{i}{\nu} \le t \le \frac{i+1}{\nu} , \; 1 \le i \le \nu-2, \\ \phi_\nu\left(\nu b\left(t - \frac{\nu-1}{\nu}\right)\right) , & \frac{\nu-1}{\nu} \le t \le 1. \end{cases}$$

The last requirement of Def. 2 is that for each face F of $\mathcal{M}$ there exist edges $e_1, \ldots, e_\nu$ satisfying (2.12) and with the final point of e_ν equal to the initial point of e_1, and such that the curve J made up of all of $e_1, e_2, \ldots, e_{\nu-1}$ and all of e_ν is a Jordan curve with $F = \text{int}(J)$[1]. In this case we call J the perimeter of F and the endpoints of the e_i the vertices on the perimeter of F. In particular each face of a mosaic is bounded.

Def. 3. Let F be a face of a mosaic $\mathcal{M}$. Close-packing F means adding an edge to $\mathcal{M}$ between any pair of vertices on the perimeter on F which are not yet adjacent.

Comment.

(iii) Without loss of generality we shall choose the interiors of new edges in the imbedding inside F when we close-pack a face F. We shall actually construct them even more carefully in Comments 2.3 (i),

---

1) If J is a Jordan curve in $\mathbb{R}^2$ then $\mathbb{R}^2 \setminus J$ consists of a bounded component denoted by $\text{int}(J)$ and an unbounded component denoted by $\text{ext}(J)$.

(iii), (v) and 2.4 (iii) when we imbed $\mathcal{G}_{p\ell}$.

Def. 4. Let $\mathcal{M}$ be a mosaic and $\mathfrak{F}$ a subset of its collection of faces. The matching pair $(\mathcal{G},\mathcal{G}^*)$ of graphs based on $(\mathcal{M},\mathfrak{F})$ is the following pair of graphs: $\mathcal{G}$ is the graph obtained from $\mathcal{M}$ by close-packing all faces in $\mathfrak{F}$. $\mathcal{G}^*$ is the graph obtained from $\mathcal{M}$ by close packing all faces not in $\mathfrak{F}$.

Comments.

(iv) If $(\mathcal{G},\mathcal{G}^*)$ is a matching pair based on $(\mathcal{M},\mathfrak{F})$ then $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{G}^*$ all have the same vertex set.

(v) If $(\mathcal{G},\mathcal{G}^*)$ is based on $(\mathcal{M},\mathfrak{F})$, then $(\mathcal{G}^*,\mathcal{G})$ is a matching pair based on $(\mathcal{M},\mathfrak{F}^*)$, where $\mathfrak{F}^*$ is the collection of faces of $\mathcal{M}$ which are not in $\mathfrak{F}$. Thus we can think of $\mathcal{G}$ as $(\mathcal{G}^*)^*$.

(vi) $\mathfrak{F} = \emptyset$ or $\mathfrak{F}$ = collection of all faces of $\mathcal{M}$ are allowed in Def. 2. Therefore any mosaic $\mathcal{M}$ equals the first graph of some matching pair - the one based on $(\mathcal{M},\emptyset)$. Compare Ex. (i) below.

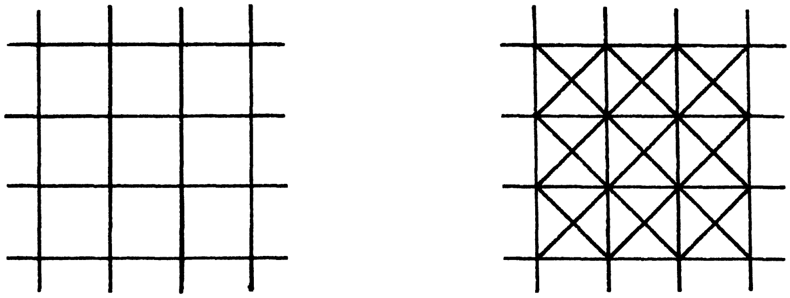
(vii) In a matching pair usually at least one of the graphs $\mathcal{G}$ or $\mathcal{G}^*$ is not planar. However, if we add the edges to $\mathcal{M}$ in conformity with Comment (iii), then an edge e of $\mathcal{G}$ and an edge e^* of $\mathcal{G}^*$ can intersect only at endpoints of these edges which are necessarily vertices of $\mathcal{M}$ (unless e and e^* coincide). Two edges e_1 and e_2 of $\mathcal{G}$ can have an intersection which is not an end-point of both of them only if e_1 and e_2 are edges whose interiors lie in the same face F of $\mathcal{M}$, which is close-packed in $\mathcal{G}$. In this situation any pair of the endpoints of e_1 and e_2 are neighbors on $\mathcal{G}$ (because F is close-packed). The same comment applies to two edges e_1^* and e_2^* of $\mathcal{G}^*$.

Examples.

(i) Let $\mathcal{M} = \mathcal{G}_0$, as defined in Ex. 2.1 (i). If we take $\mathfrak{F} = \emptyset$, then the complementary collection of faces, $\mathfrak{F}^*$, consists of all squares into which the plane is divided by the lines $x(1)=k$ and $x(2)=\ell$ $(k,\ell \in \mathbb{Z})$. The matching pair $(\mathcal{G},\mathcal{G}^*)$ based on $(\mathcal{M},\mathfrak{F})$ in this case is described by $\mathcal{G} = \mathcal{G}_0 = \mathcal{M}$ and $\mathcal{G}^*$ is the graph with vertex set $\mathbb{Z}^2$ while (i_1,i_2) and (j_1,j_2) are adjacent on $\mathcal{G}^*$ iff (2.6) holds or

$$(2.13) \qquad |i_1-j_1| = |i_2-j_2| = 1.$$

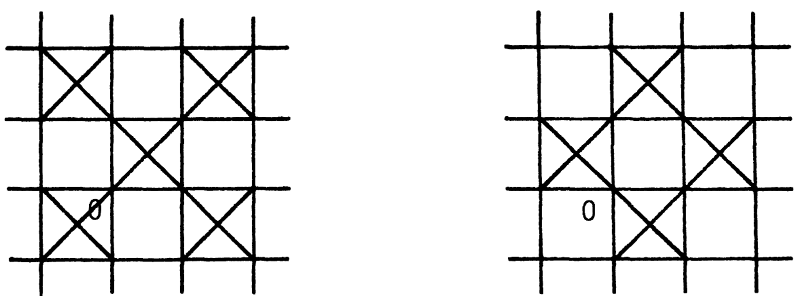
The graphs $\mathcal{G}$ and $\mathcal{G}^*$ are illustrated in Fig. 2.8. $\mathcal{G}^*$ is obtained by adding all "diagonals" to $\mathcal{G}_0$.



$\mathcal{G} = \mathcal{G}_0 = \mathcal{M}$ $\qquad$ $\mathcal{G}^*$

Figure 2.8

(ii) Again take $\mathcal{M} = \mathcal{G}_0$ as above, but this time take $\mathcal{F}$ to be the collection of unit squares $(i_1, i_1+1) \times (i_2, i_2+1)$ with i_1+i_2 even. $\mathcal{F}^*$ will consist of the unit squares $(i_1, i_1+1) \times (i_2, i_2+1)$ with i_1+i_2 odd. $\mathcal{G}$ will be the graph of Fig. 2.2. $\mathcal{G}^*$ will be a similar graph but now with the diagonals in the set of unit squares which is empty in $\mathcal{G}$. (The formal description is as for $\mathcal{G}$ in Ex. 2.1 (ii) but with odd and even interchanged in (2.7).) Fig. 2.9 shows a picture of the matching pair $(\mathcal{G}, \mathcal{G}^*)$ in this example.



$\mathcal{G}$ $\qquad$ Figure 2.9 $\qquad$ $\mathcal{G}^*$

Clearly $\mathcal{G}$ and $\mathcal{G}^*$ are isomorphic as graphs. Such a pair is called <u>self-matching</u>.

(iii) Any triangular face is already close-packed. Thus if $\mathcal{M}$ has only triangular faces, then for every choice of $\mathcal{F}$ the matching pair based on $(\mathcal{M}, \mathcal{F})$ is $\mathcal{G} = \mathcal{M}$, $\mathcal{G}^* = \mathcal{M}$. Such a pair is again self-matching. An example of this situation is $\mathcal{M} = \mathcal{T}$, the triangular lattice of ex. 2.1 (iii). Another example for such an $\mathcal{M}$ is the centered quadratic lattice. Its vertex set is $\mathbb{Z}^2 \cup \{(i_1+\frac{1}{2}, i_2+\frac{1}{2}): i_1, i_2 \in \mathbb{Z}\}$.

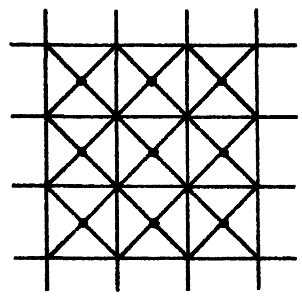


Figure 2.10 The centered quadratic lattice

Two vertices (i_1,i_2), (j_1,j_2) with $i_r, j_r \in \mathbb{Z}$ are adjacent iff (2.6) holds; (i_1,i_2) and $(j_1+\frac{1}{2}, j_2+\frac{1}{2})$ are adjacent iff

(2.14) $\quad i_1 = j_1$ or j_1+1 and $i_2 = j_2$ or j_2+1.

This graph is not the same as $\mathcal{G}^*$ in Ex. 2.2 (i) because the present $\mathcal{M}$ has vertices at the centers of all the squares, while $\mathcal{G}^*$ of Ex. 2.2 (i) does not.

(iv) The following example illustrates the gain of generality of allowing multiple edges between the same vertices. The vertex set is $\mathbb{Z}^2 \cup \{i_1+\frac{1}{2}, i_2\}$ (i.e., we increase the vertex set $\mathbb{Z}^2$ by adding a vertex in the middle of each horizontal link). The following will be the edges in $\mathcal{M}$: any vertical link between (i_1,i_2) and $(i_1, 1_2+1)$; the horizontal links between (i_1,i_2) and $(i_1 \pm \frac{1}{2}, i_2)$; and two extra edges between (i_1,i_2) and (i_1+1,i_2), one running in each of the squares above and below the line $x(2)=i_2$ (see Fig. 2.11). The vertices $(i_1+\frac{1}{2}, i_2)$ belong to two triangular faces. If we take for $\mathcal{F}$ the collection of all other faces, i.e., the quadrilaterals, then $\mathcal{G}^* = \mathcal{M}$, while $\mathcal{G}$ gains the diagonals in each of the quadrilaterals. If we remove one of the edges from (i_1,i_2) to (i_1+1,i_2) from $\mathcal{G}$, to obtain a graph without multiple edges, then the resulting graph, $\mathcal{G}'$ say, contains the configurations of Fig. 2.12.

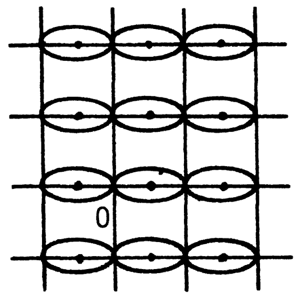


Figure 2.11 Illustration of multiple edges. $\mathcal{M}$

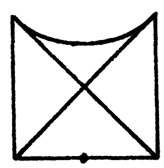


Figure 2.12

Two edges intersect in their interior, but they do not seem to belong to a close-packed face of a mosaic. Thus it seems impossible to view $\mathcal{G}'$ as one of a matching pair. Nevertheless, for the site-percolation problem $\mathcal{G}'$ is equivalent to $\mathcal{G}$ (see Ch. 3). It therefore seems of some use to allow multiple edges.

2.3 Planar modifications of matching pairs.

For many proofs it is a great convenience to work with planar graphs. The main advantage is that a self-avoiding path $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ on a planar graph truly has no self intersections. I.e., even though "r is self-avoiding" in our terminology only means that all v_i are different, on a planar graph this also implies that e_i cannot intersect e_j for $j > i$ unless $j = i+1$, and then $e_i \cap e_j = \{ v_i \}$ = the common endpoint of e_i and e_j. This is obvious from the fact that on a planar graph two distinct edges can only intersect in a common endpoint. In view of these considerations we introduce planar modifications $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ of a matching pair $\mathcal{G}$ and $\mathcal{G}^*$, as well as planar modification $\mathcal{M}_{p\ell}$ of a mosaic $\mathcal{M}$. For our purposes $\mathcal{G}$ and $\mathcal{G}_{p\ell}$ (as well as $\mathcal{G}^*$ and $\mathcal{G}^*_{p\ell}$) will be practically interchangeable (see Lemma 2.1). Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair based on $(\mathcal{M},\mathcal{F})$. We then construct $\mathcal{G}_{p\ell}$ as follows: Its vertex set is the vertex set of $\mathcal{G}$ plus one additional vertex in each face F of $\mathcal{F}$. The added vertex inside F will be called the central vertex of F. Two vertices v and w of $\mathcal{G}_{p\ell}$ will be adjacent on $\mathcal{G}_{p\ell}$ iff v and w are adjacent on $\mathcal{M}$, or if one of them is the central vertex of some $F \in \mathcal{F}$ and the other is on the perimeter of the same face F. The edge set of $\mathcal{G}_{p\ell}$ therefore consists of the edge set of $\mathcal{M}$ plus, for each $F \in \mathcal{F}$, edges between the central vertex of F and all the vertices on the perimeter of $\mathcal{F}$.

Comments.

(i) In order to show that $\mathcal{G}_{p\ell}$ is indeed planar we give an imbedding in $\mathbb{R}^2$ "explicitly". Let $F \in \mathcal{F}$ be a face with perimeter J. There then exists a homeomorphism ψ from $\overline{F} := F \cup J$ onto the closed unit disc (by Theorem VI.17.1 of Newman (1951) or by the Riemann mapping theorem, Hille (1962), Theorem 17.5.3). Let v_i, $1 \leq i \leq \nu$ be the vertices of $\mathcal{G}$ (or $\mathcal{M}$) on J and $w_i = \psi(v_i)$, $1 \leq i \leq \nu$ their images on the unit circle. Then place the central vertex of F at $\psi^{-1}(0)$, and take for the edge from the central vertex to v_i the inverse image under ψ of the ray from 0 to w_i. We can use this construction at the same time to obtain a pleasant imbedding for $\mathcal{G}$ itself. We merely take for the edge between v_i and v_j, two non-adjacent vertices on J, the inverse image of the line segment from w_i to w_j. This gives us a simultaneous imbedding of $\mathcal{G}$ and $\mathcal{G}_{p\ell}$ such that the edge of $\mathcal{G}$ from v_i to v_j intersects the edges of $\mathcal{G}_{p\ell}$ from v_i and v_j to the central vertex only in v_i and v_j. Also if e_1 and e_2 are two edges of $\mathcal{G}$ in the face F, with endpoints v_1, v_2 and v_3, v_4, respectively, then e_1 can intersect e_2 in a point different from v_3, v_4 only if the four points v_1-v_4 are distinct, and v_1, v_2 separate v_3, v_4 on J. In other words, each of the two arcs of J between v_3 and v_4 must contain one of v_1 and v_2.

(ii) Note that we inserted a central vertex in every face $F \in \mathcal{F}$, even if F is a triangle, i.e., bounded by three edges, or a "lens", i.e., bounded by two different edges with the same pair of endpoints. Such faces contain no extra edges in $\mathcal{G}$ when compared to $\mathcal{M}$, but these faces become different after close-packing (compare Ex.2.3 (iii) below).//

$\mathcal{G}^*_{p\ell}$ is defined and constructed in exactly the same way as $\mathcal{G}_{p\ell}$ above; we merely have to replace $\mathcal{G}$ by $\mathcal{G}^*$ and $\mathcal{F}$ by $\mathcal{F}^*$ throughout. In particular $\mathcal{G}^*_{p\ell}$ has only central vertices in faces of $\mathcal{F}^*$, but not in faces of $\mathcal{F}$. A more explicit notation would be $(\mathcal{G}^*)_{p\ell}$. This is not the same as $(\mathcal{G}_{p\ell})^*$, the latter being the second graph of the matching pair $(\mathcal{G}_{p\ell},(\mathcal{G}_{p\ell})^*)$ based on $(\mathcal{G}_{p\ell},\emptyset)$. In these notes we shall never use $(\mathcal{G}_{p\ell})^*$ and $\mathcal{G}^*_{p\ell}$ will always stand for $(\mathcal{G}^*)_{p\ell}$.

$\mathcal{M}_{p\ell}$ is the graph whose vertex (edge) sets is the union of the vertex (edge) sets of $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$. Thus $\mathcal{M}_{p\ell}$ has a central vertex added in each face of $\mathcal{M}$.

Comments.

(iii) If $\mathcal{G}$ is periodic, we want to take $\mathcal{G}_{p\ell}$ also periodic. To see that this can be done observe first that if F is any face of $\mathcal{M}$ and $x \in \mathbb{R}^2$ a point in F then $x+k_1e_1+k_2e_2 \notin F$ if $k_1,k_2 \in \mathbb{Z}$, not both zero. For, otherwise, there would exist a continuous path from x to $x+k_1e_1+k_2e_2$ which does not intersect any edge of $\mathcal{M}$. Extending this path periodically would give an unbounded path in F, so that the face F of $\mathcal{M}$ would have to be unbounded. But all faces of $\mathcal{M}$ are the interiors of Jordan curves, and hence bounded. This proves the observation. It follows that for any face $F \in \mathcal{F}$, all the faces $F+k_1e_1+k_2e_2$, $k_1,k_2 \in \mathbb{Z}$, are pairwise disjoint. Since $\mathcal{G}$ is periodic, $F \in \mathcal{F}$ implies that this whole class belongs to $\mathcal{F}$. As a result $\mathcal{F}$ can be written as a disjoint union of classes $\mathcal{F}_i$, each $\mathcal{F}_i$ of the form $\{F_i+k_1e_1+k_2e_2: k_1,k_2 \in \mathbb{Z}\}$ and all faces in one $\mathcal{F}_i$ disjoint from each other. If we now add a central vertex in F_i, and edges from this central vertex to the vertices on the perimeter of F_i, then we can repeat this construction periodically in every face $F_i+k_1e_1+k_2e_2$. Since all these faces are disjoint these constructons do not interfere with each other and the resulting $\mathcal{G}_{p\ell}$ is periodic.

(iv) Two central vertices are never adjacent. This holds on $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$.

(v) The imbedding of Comments (i) and (iii) can be extended to give a simultaneous imbedding of $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$. An edge e of $\mathcal{G}_{p\ell}$ and an edge e^* of $\mathcal{G}^*_{p\ell}$ can intersect only in a common endpoint, which is necessarily a vertex of $\mathcal{M}$, unless e and e^* coincide (compare Comment 2.2 (vii)). We can even imbed $\mathcal{G}$, $\mathcal{G}^*$, $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$ simultaneously. In this case any edge e of $\mathcal{G}$ belongs to the closure $\overline{F}$ of some face F of $\mathcal{F}$. (See Comment 2.2 (iii).) On the other hand, any edge e^* of $\mathcal{G}_{p\ell}$ will either be also an edge of $\mathcal{G}$ or $\mathring{e}^* \subset F^*$ for some face $F^* \notin \mathcal{F}$. Therefore an edge e of $\mathcal{G}$ and an edge e^* of $\mathcal{G}_{p\ell}$ which do not coincide can again intersect only in a common endpoint which is a vertex of $\mathcal{M}$.

(vi) Any face $F \in \mathcal{F}$ of $\mathcal{M}$ becomes triangulated in $\mathcal{G}_{p\ell}$. Similarly for $F \in \mathcal{F}^*$ in $\mathcal{G}^*_{p\ell}$. All faces of $\mathcal{M}_{p\ell}$ are "triangles", i.e., are bounded by a Jordan curve made up from three edges of $\mathcal{M}_{p\ell}$.

Examples.

(i) Let $(\mathcal{M},\mathcal{F}) = (\mathcal{G}_0,\emptyset)$ and $(\mathcal{G},\mathcal{G}^*)$ the matching pair based on this as in Ex. 2.2 (i). Then $\mathcal{G}_{p\ell} = \mathcal{G} = \mathcal{G}_0 = \mathcal{M}$ while $\mathcal{G}^*_{p\ell} = \mathcal{M}_{p\ell}$ is the centered quadratic lattice of Ex. 2.2 (iii).

(ii) For $\mathcal{M}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{G}^*$ as in Ex. 2.2 (ii) $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ are obtained by adding a vertex to $\mathcal{M}$ at each point $(i_1+\frac{1}{2}, i_2+\frac{1}{2})$ with i_1+i_2 = even and odd, respectively, and connecting it by an edge to the vertices (i_1,i_2), (i_1+1,i_2), (i_1+1,i_2+1), (i_1,i_2+1).

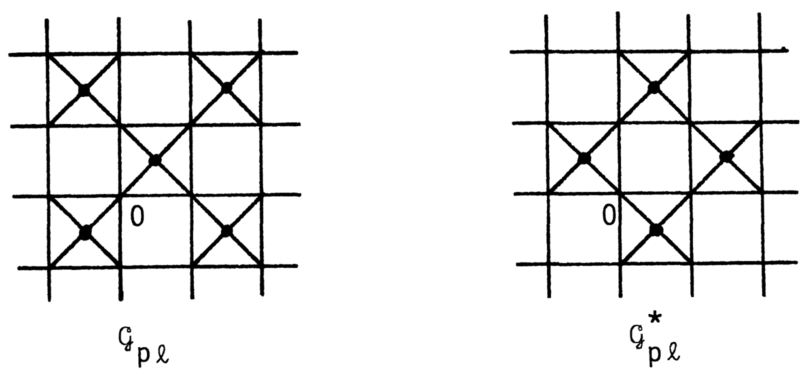


Figure 2.13

(iii) If $\mathcal{M} = \mathcal{T}$ (see Ex. 2.1 (iii)), the triangular lattice, and $\mathcal{F} = \emptyset$, then $\mathcal{G}_{p\ell} = \mathcal{G} = \mathcal{T}$, but $\mathcal{G}^*_{p\ell} = \mathcal{M}_{p\ell}$ has a central vertex placed in each triangle, and this is connected by an edge to each of the three vertices of the triangle. ///

We must now turn to the relationship between $\mathcal{G}$ and $\mathcal{G}_{p\ell}$ - as well as $\mathcal{G}^*$ and $\mathcal{G}^*_{p\ell}$ - in the context of percolation. For reasons to be explained in Sect. 3.1 we restrict ourselves to site percolation. <u>In the remainder of this section</u> $(\mathcal{G},\mathcal{G}^*)$ <u>is a matching pair, based on</u> $(\mathcal{M},\mathcal{F})$.

<u>Def. 5</u>. An <u>occupancy configuration</u> for $\mathcal{M}$ or $\mathcal{G}$ or $\mathcal{G}^*$ is a map ω from their common vertex set into $\{-1,+1\}$. ///

Usually we call a vertex v with $\omega(v) = +1$ occupied, and with $\omega(v) = -1$ vacant. Thus, an occupancy configuration is merely a partition of the vertices into occupied and vacant ones. We shall always extend such a configuration to an occupancy configuration for $\mathcal{M}_{p\ell}$ or $\mathcal{G}_{p\ell}$ or $\mathcal{G}^*_{p\ell}$ (in obvious terminology) by setting

(2.15) $\omega(v) = +1$ for every central vertex v of an $F \varepsilon \mathcal{F}$,

(2.16) $\omega(v) = -1$ for every central vertex v of an $F \varepsilon \mathcal{F}^*$.

Thus, <u>we always take central vertices of</u> $\mathcal{G}_{p\ell}$ <u>occupied and of</u> $\mathcal{G}^*_{p\ell}$ <u>vacant</u>. (We only make an exception to this rule in Sect. 10.2 and 3, where the exception will be explicitly pointed out.)

Definitions 6 and 7 below refer to a fixed occupancy configuration ω, but usually we suppress the dependence on ω from the notation.

<u>Def. 6</u>. An <u>occupied path</u> on $\mathcal{G}(\mathcal{G}_{p\ell})$ is a path on $\mathcal{G}(\mathcal{G}_{p\ell})$ all of whose vertices are occupied. A <u>vacant path</u> on $\mathcal{G}^*$ $(\mathcal{G}^*_{p\ell})$ is a path on $\mathcal{G}^*(\mathcal{G}^*_{p\ell})$ all of whose vertices are vacant. ///

The following lemma will allow us to go back and forth between paths on $\mathcal{G}$ and $\mathcal{G}_{p\ell}$ (or $\mathcal{G}^*$ and $\mathcal{G}^*_{p\ell}$).

<u>Lemma 2.1a</u>. <u>Let</u> $r = (v_0, e_1, v_1, \dots, e_\nu, v_\nu)$ <u>be an occupied path on</u> $\mathcal{G}_{p\ell}$. <u>Then there exists an occupied path</u> $\tilde{r}$ <u>on</u> $\mathcal{G}$ <u>whose vertices are exactly the non-central vertices of</u> r, <u>and they occur in the same order in</u> $\tilde{r}$ <u>as in</u> r. <u>Moreover, if</u>

(2.17) diameter (e) $\leq \Lambda$ for all edges e of $\mathcal{G}$ and $\mathcal{G}_{p\ell}$

<u>then</u>

(2.18) for each point x of $\tilde{r}$ there exists a vertex y of r with $|x-y| \leq \Lambda$.

<u>Lemma 2.1b</u>. <u>Let</u> $\tilde{r} = (\tilde{v}_0, \tilde{e}_1, \dots, \tilde{e}_\rho, \tilde{v}_\rho)$ <u>be an occupied path on</u> $\mathcal{G}$. <u>Then there exists an occupied path</u> r <u>on</u> $\mathcal{G}_{p\ell}$ <u>(as well as on</u> $\mathcal{M}_{p\ell}$<u>)</u> <u>from</u> $\tilde{v}_0$ <u>to</u> $\tilde{v}_\rho$. <u>The non-central vertices of</u> r <u>form a subset of the vertices of</u> $\tilde{r}$, <u>and occur in the same order in</u> r <u>as in</u> $\tilde{r}$. <u>Moreover if</u> (2.17) <u>holds, then</u>

(2.19) for each point y of r there exists a vertex x of $\tilde{r}$ such that $|x-y| \leq \Lambda$.

Proof of Lemma 2.1a. Let $v_{i_0}, v_{i_1}, \dots, v_{i_\rho}$ with $i_0 < i_1 < \dots < i_\rho$ be all the vertices of r which are not central vertices of $\mathcal{G}_{p\ell}$. Since by construction $\mathcal{G}_{p\ell}$ has no edges between two central vertices, there cannot be two successive central vertices in r (cf. Comment 2.3 (iv)). Consequently, $i_0 \le 1$, $i_{j+1} - i_j \le 2$ and $i_\rho \ge \nu - 1$. If $i_{j+1} = i_j + 1$ then we simply connect v_{i_j} and $v_{i_{j+1}} = v_{i_j+1}$ by the edge e_{i_j+1} from r. In this case we take $\tilde{e}_{j+1} = e_{i_j+1}$. If $i_{j+1} = i_j + 2$, then v_{i_j+1} is a central vertex of some face F of $\mathcal{M}$ and v_{i_j} and v_{i_j+2} are both vertices of $\mathcal{G}$ on the perimeter of F. Moreover the face F must be close-packed in $\mathcal{G}$, so that there is an edge $\tilde{e}_{j+1}$ of $\mathcal{G}$ contained in the closure of F which connects v_{i_j} and $v_{i_{j+1}} = v_{i_j+2}$. We define $\tilde{v}_j = v_{i_j}$ and take $r = (\tilde{v}_0, \tilde{e}_1, \dots, \tilde{e}_\rho, \tilde{v}_\rho)$. It is easy to see that $\tilde{r}$ satisfies the requirements of the lemma. We merely remark that $\tilde{r}$ is self-avoiding (recall our convention that a path should be self-avoiding). To see this note that the $\tilde{v}_j$ form a subset of the vertices of the self-avoiding path r, and therefore are distinct. The vertices of $\tilde{r}$ are the non-central vertices of r, in their original order. $\tilde{r}$ is automatically occupied, since all its vertices are also vertices of the occupied path r. Finally, (2.17) implies (2.18) because any point x of $\tilde{r}$ lies on some edge $\tilde{e}_i$ of $\tilde{r}$, and by virtue of (2.17) lies within Λ from $\tilde{v}_i$, which is also a vertex of r.

Proof of Lemma 2.1b. This proof is almost the reverse of that of Lemma 2.1a. We now insert central vertices whenever necessary. More precisely, if $\tilde{e}_i$ is an edge of $\mathcal{G}$, but not of $\mathcal{G}_{p\ell}$, then its interior must lie in a close-packed face F and its endpoints $\tilde{v}_{i-1}$ and $\tilde{v}_i$ must lie on the perimeter of F. Let c be the central vertex of F and e', e'' the edges of $\mathcal{G}_{p\ell}$ between $\tilde{v}_{i-1}$ and c, and between c and $\tilde{v}_i$, respectively. We now replace the edge $\tilde{e}_i$ by e', c, e''. If $\tilde{e}_i$ is already an edge of $\mathcal{G}_{p\ell}$, then of course it need not be replaced. We make all the necessary replacements for $i = 1, \dots, \rho$, and denote the resulting sequence of vertices and edges of $\mathcal{G}_{p\ell}$ by $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ The v_j consists of the $\tilde{v}_i$ in their original order, with some central vertices of $\mathcal{G}_{p\ell}$ interpolated and $v_0 = \tilde{v}_0$, $v_\nu = \tilde{v}_\rho$. All v_i are automatically occupied by virtue of (2.15) and the fact that $\tilde{r}$ was an occupied path. If r itself is not self-avoiding we make it self-avoiding by loop-removal without changing its first or last point, as described in Sect. 2.1. The resulting path satisfies all requirements.

For (2.19), observe that each edge in the final path has at least one endpoint which is not a central vertex, and hence also belongs to $\tilde{r}$. □

Def. 7. $W(v) = W(v,\omega)$, the occupied cluster or occupied component of v on $\mathcal{G}$ is the union of all edges and vertices of $\mathcal{G}$ which belong to an occupied path on $\mathcal{G}$ with initial point v. ///

v is assumed to be a vertex of $\mathcal{G}$ in Def. 7. If v is vacant $W(v) = \phi$. If v is occupied, but all its neighbors are vacant, then $W(v) = \{v\}$.

We define the occupied cluster $W_{p\ell}(v) = W_{p\ell}(v,\omega)$ of v on $\mathcal{G}_{p\ell}$ by replacing $\mathcal{G}$ by $\mathcal{G}_{p\ell}$ in the above definition. No confusion with the occupied cluster of v on $\mathcal{M}_{p\ell}$ can arise because the latter equals $W_{p\ell}(v)$. In fact, by virtue of (2.16), an occupied path on $\mathcal{M}_{p\ell}$ cannot contain any central vertices of $\mathcal{G}^*_{p\ell}$, and therefore is an occupied path on $\mathcal{G}_{p\ell}$ itself. Lemma 2.1 therefore has the following corollary.

Corollary 2.1. For any vertex v of $\mathcal{G}$

(2.20) $W_{p\ell}(v) = W(v) \cup \{$all edges of $\mathcal{G}_{p\ell}$ from a central vertex of $\mathcal{G}_{p\ell}$ to some vertex $w \in W(v)\}$.

Consequently, if (2.17) holds, then

(2.21) $W \subset W_{p\ell} \subset \{x: |x-w| \leq \Lambda$ for some $w \in W\}$.

Proof: If w is a non-central vertex of $\mathcal{G}_{p\ell}$ which can be connected to v by an occupied path on $\mathcal{G}_{p\ell}$, then it is also connected to v by an occupied path on $\mathcal{G}$, and vice versa, by virtue of Lemma 2.1a and 2.1b, respectively. Thus the non-central vertices of $W_{p\ell}$ are precisely the vertices of W. Since central vertices only have non-central neighbors on $\mathcal{G}_{p\ell}$ (see Comment 2.3 (iv)) one easily sees that the central vertices of $W_{p\ell}$ are precisely those vertices which are adjacent to some (non-central) vertex of W (recall that v is a vertex of $\mathcal{G}$, hence non-central) (2.20) is immediate from this, while (2.21) follows from (2.17) and (2.20). □

Remark.

One can define a planar modification for certain graphs $\mathcal{G}$ which are not necessarily one of a matching pair. It seems that many percolation results will go through for such graphs. Specifically, let $\mathcal{M}$ be a mosaic and F a face of $\mathcal{M}$ with perimeter J. Let $G_1, \ldots, G_k$ be pairwise disjoint sets of vertices on J, such that no two vertices of G_i separate two of G_j on J, when $i \neq j$. Assume that $\mathcal{G}$ is formed by adding an edge in $\overline{F}$ between any pair of vertices which belong to the same G_i, and that these edges are added such that an edge between two vertices of G_i does not intersect an edge between two vertices of G_j for $i \neq j$. For example if F is an octagon as in Fig. 2.14a, then G_1 (G_2) might be the top (bottom) four vertices, and $\mathcal{G}$ would have

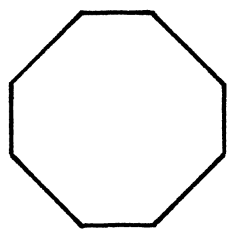


Figure 2.14a $\mathcal{M}$

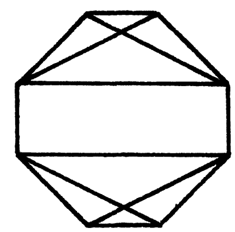


Figure 2.14b $\mathcal{G}$

edges in $\overline{F}$ as indicated in Fig. 2.14b. Such edges could be added in many faces. To form a planar modification - call it $\mathcal{G}_{p\ell}$ again - one would now insert one central vertex v_i in F for each G_i. In $\mathcal{G}_{p\ell}$ v_i would be connected by an edge to each vertex in G_i but to no other vertices. For the situation illustrated in Fig. 2.14a and b we would end up with the situation of Fig. 2.14c. If we take all central vertices

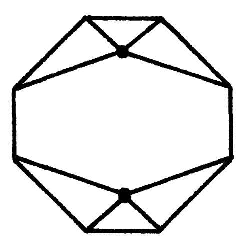


Figure 2.14c $\mathcal{G}_{p\ell}$ corresponding to the $\mathcal{G}$ of Figure 2.14b.

occupied as in (2.15), then Lemmas 2.1a and b and Cor. 2.1 remain valid (with only trivial changes in their proofs). As in the case when $\mathcal{G}$ is one of a matching pair this will allow us to reduce site-percolation problems on $\mathcal{G}$ to equivalent ones on $\mathcal{G}_{p\ell}$. Even though there is no obvious analogue of $\mathcal{G}^*$ for the more general graphs of this remark, we can apply much of the succeeding directly to $\mathcal{G}_{p\ell}$. Indeed $\mathcal{G}_{p\ell}$ is planar, and a mosaic. Thus $\mathcal{G}_{p\ell}$ is one of a pair of matching graphs, based on $(\mathcal{G}_{p\ell},\phi)$ (see Comment 2.2 (vi)). Moreover, if we view $\mathcal{G}_{p\ell}$ as based on $(\mathcal{G}_{p\ell},\phi)$, then $(\mathcal{G}_{p\ell})_{p\ell} = \mathcal{G}_{p\ell}$. Thus results for one of a matching pair of graphs apply to $\mathcal{G}_{p\ell}$.

2.4 Separation theorems and related point-set topological results.

In this section we formulate some purely graph-theoretical, or point-set topological nature which will play a fundamental role. Their contents are easily acceptable intuitively, but their proofs are somewhat involved. For this reason we postpone the proof of Prop. 2.1-2.3 to the Appendix. In this section $(\mathcal{G},\mathcal{G}^*)$ is again a matching pair of graphs, based on $(\mathcal{M},\mathcal{F})$. First we need the definition of the boundary of a set on a graph.

Def. 8. Let A be a subset of a graph $\mathcal{H}$. Its boundary on $\mathcal{H}$ is the set

$$\partial A = \{v: v \text{ a vertex of } \mathcal{H} \text{ outside } A, \text{ but there exists a vertex } w \in A \text{ such that } v \overset{\mathcal{H}}{\sim} w\}.$$

The notation ∂A does not indicate any dependence on $\mathcal{H}$. However, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two graphs such that A can be viewed as a subset of both of them then the boundary of A may be different on $\mathcal{H}_1$ and on $\mathcal{H}_2$. If necessary we shall indicate on which graph the boundary is to be taken. Often it will be clear from knowing the set A which boundary is intended. E.g. $W(v)$ is defined as a cluster on $\mathcal{G}$, and correspondingly ∂W will always mean the boundary of W on $\mathcal{G}$. On the other hand $\partial W_{p\ell}$ means the boundary of $W_{p\ell}$ on $\mathcal{M}_{p\ell}$, and not on $\mathcal{G}_{p\ell}$, in Prop. 1.

Def. 9. Let $\mathcal{H}$ be a graph imbedded in $\mathbb{R}^2$ and $A \subset \mathbb{R}^2$. A circuit on $\mathcal{H}$ surrounding A is a Jordan curve on $\mathcal{H}$ made up of edges of $\mathcal{H}$ which contains A in its interior (cf. Comment 2.2(ii)). We call the circuit occupied (vacant) if all vertices of $\mathcal{H}$ on the circuit are occupied (vacant).

The following proposition is a version of Theorem 4 in Whitney (1933); it is of fundamental importance in the development of percolation theory.

Proposition 2.1. *Let* $\partial W_{p\ell}(v)$ *be the boundary of* $W_{p\ell}(v)$ on $\mathcal{M}_{p\ell}$. *If*

$W_{p\ell}(v)$ <u>is non-empty and bounded and (2.3)-(2.5) hold with</u> $\mathcal{G}$ <u>replaced by</u> $\mathcal{M}$, <u>then there exists a vacant circuit</u> $J_{p\ell}$ <u>on</u> $\mathcal{M}_{p\ell}$ <u>surrounding</u> $W_{p\ell}(v)$ <u>and such that all vertices of</u> $\mathcal{M}_{p\ell}$ <u>on</u> $J_{p\ell}$ <u>belong to</u> $\partial W_{p\ell}(v)$.

<u>Corollary 2.2</u>. <u>If</u> $W(v)$ <u>is non-empty and bounded and</u> (2.3)-(2.5) <u>hold</u>, <u>then there exists a vacant circuit</u> J <u>on</u> $\mathcal{G}^*$ <u>surrounding</u> $W(v)$.

The proof of this corollary is also in the appendix. We stress that $W(v)$ is a subset of $\mathcal{G}$ while the surrounding circuit is on $\mathcal{G}^*$. It is easy to see that there does not have to exist a circuit on $\mathcal{G}$ itself surrounding W. E.g., if $(\mathcal{G},\mathcal{G}^*)$ is the matching pair of ex.2.2(i) and

$$\omega(v) = \begin{cases} +1 & \text{if } v(1) = v(2) \ (v = (v(1),v(2))) \\ -1 & \text{otherwise,} \end{cases}$$

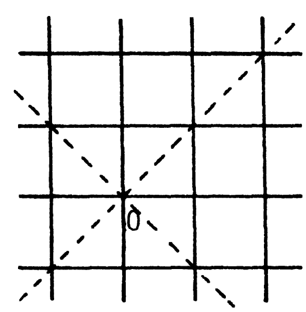


Figure 2.15 v is occupied iff v lies on one of the dashed 45° lines.

then $W(0) = \{0\}$, but no vacant path on $\mathcal{G}$ surrounds the origin.

<u>Proposition 2.2</u>. <u>Let</u> J <u>be a Jordan curve on</u> $\mathcal{M}$ (<u>and hence also on</u> $\mathcal{G}$ <u>and on</u> $\mathcal{G}^*$) <u>which consists of four closed arcs</u> A_1,A_2,A_3,A_4 <u>with disjoint interiors, and such that</u> A_1 <u>and</u> A_3 <u>each contain at least one vertex of</u> $\mathcal{M}$. <u>Assume that one meets these arcs in the order</u> A_1,A_2,A_3,A_4 <u>as one traverses</u> J <u>in one direction</u>. <u>Then there exists a path</u> r <u>on</u> $\mathcal{G}$ <u>inside</u> $\bar{J} := J \cup \text{int}(J_1)$ <u>from a vertex on</u> A_1 <u>to a vertex on</u> A_3, <u>and with all vertices of</u> r <u>in</u> $\bar{J}\backslash A_1 \cup A_3$ <u>occupied, if and only if there does not exist a vacant path</u> r^* <u>on</u> $\mathcal{G}^*$ <u>inside</u> $\bar{J}\backslash A_1 \cup A_3$ <u>from</u>

<u>a vertex of</u> $\mathring{A}_2$ <u>to a vertex of</u> $\mathring{A}_4$.[1)]

The next proposition is a cornerstone of the development in these notes. Some form of it has been used by many authors. In its simplest form it says that among all occupied paths connecting the left and right edge of a rectangle there exists a unique lowest one. This has often been taken for granted. Harris (1960) quotes a general theorem from topology to prove this for bond-percolation on $\mathbb{Z}^2$. We do, however, need a more general result, and this is not valid on all graphs. We therefore give a proof of Prop. 2.3 in the Appendix, which closely follows Lemma 1 of Kesten (1980a). An examination of the proof will show that it is crucial that $\mathcal{G}_{p\ell}$ is planar.

Some preparation concerning symmetry axes of a graph, and a partial ordering of paths is needed first.

<u>Def. 10</u>. Let $\mathcal{H}$ be a graph imbedded in $\mathbb{R}^2$. The line $L: x(1) = a$ is an <u>axis of symmetry</u> for $\mathcal{H}$ if $\mathcal{H}$ is invariant under the reflection of L which takes $y = (y(1),y(2))$ to $(2a-y(1),y(2))$. Similarly if L is a vertical line of the form $x(2) = a$.

<u>Comments</u>.

(i) L is an axis of symmetry for $\mathcal{H}$ iff the image under the imbedding of each vertex and edge of $\mathcal{H}$ goes over into the image under the imbedding of a vertex and edge, respectively, under reflection in L. It would therefore be more accurate, but also more cumbersome to call L an axis of symmetry for the imbedding of $\mathcal{H}$, rather than for $\mathcal{H}$.

As we pointed out after Def. 2.1 of a periodic graph, percolation problems depend only on the abstract structure of the graphs, and not on their imbedding. Just as one can sometimes change an imbedding to obtain a periodic one, one can sometimes change an imbedding to make one of the coordinate-axes an axis of symmetry. E.g. neither of the coordinate-axes is an axis of symmetry for the imbedding of Fig. 2.5 for the triangular lattice, while both of them are for the imbedding of Fig. 2.4. Even though we require in several theorems that the graph is imbedded periodically and with a coordinate axis as symmetry axis, what really counts is that the graph <u>can be</u> imbedded such that it has these properties.

(ii) Assume L is an axis of symmetry for $\mathcal{G}_{p\ell}$ and e an edge

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1) When A is an arc we use $\mathring{A}$ to denote A minus its endpoints.

of $\mathcal{G}_{p\ell}$ which intersects L in a point m. Denote by $\tilde{e}$ the reflection of e in L. Then $\tilde{e}$ is also an edge of $\mathcal{G}_{p\ell}$ which intersects L in m. We shall show that exactly one of the four following cases must obtain:

(a) e lies on L,

(b) e has both endpoints, but no other points on L, and these points on L are the only common points of e and $\tilde{e}$,

(c) e has exactly one endpoint, but no other points on L, and this point on L is the only common point of e and $\tilde{e}$,

(d) e intersects L only in one point m which is not an endpoint, e is symmetric with respect to L, i.e., e coincides with $\tilde{e}$, and m is the midpoint of e.

To see this assume first that e intersects L also in another point $m' \neq m$. Then e intersects $\tilde{e}$ in m and $m' \neq m$. Then either m and m' are common endpoints of e and $\tilde{e}$ and they have no further points in common (case (b)) or e and $\tilde{e}$ also have an interior point in common, in which case they coincide and case (a) obtains (recall that $\mathcal{G}_{p\ell}$ is planar and e is a simple arc). Next consider the situation where e intersects L only in m. If m is an endpoint of e then we are in case (c), because e and $\tilde{e}$ must lie on opposite sides of L. Finally if the common point m of e and $\tilde{e}$ is an interior point of either one of them then they must again coincide, and case (d) obtains.

A good illustration of this situation is provided by the triangular lattice of Fig. 2.4. The lines $x(1) = k$, $k \in \mathbb{Z}$ are axes of symmetry; half the horizontal edges are in case (d), while the other half and all non-horizontal edges are in case (c).

(iii) In most of our theorems we shall deal with a matching pair of periodic graphs $(\mathcal{G},\mathcal{G}^*)$ based on $(\mathcal{M},\mathcal{F})$ for which $x(1) = 0$ and/or $x(2) = 0$ is an axis of symmetry. In the proofs we shall work with the planar modifications $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$, and it will be necessary that these graphs too have $x(1) = 0$ and/or $x(2) = 0$ as axis of symmetry, in addition to the properties of Comments 2.3(i),(iii) and (v). This can be achieved as follows. If $L_1: x(1) = 0$ is an axis of symmetry and $F \in \mathcal{F}$ is a face of $\mathcal{M}$ which intersects L_1, then $\overline{F}$ is symmetric with respect to L_1. We then choose the homeomorphism ψ on F of Comment 2.3 (i) such that $\psi(\tilde{x}) = \widetilde{\psi(x)}$, where $\tilde{x}$ is the reflection of x in L_1. A map ψ with this symmetry property obviously exists; simply construct ψ on $\overline{F} \cap \{x(1) \geq 0\}$ such that $\overline{F} \cap \{x(1) = 0\}$ is mapped into

$\{x(1) = 0\}$ and then reflect in L_1 (see Newman (1951), ex. VI.18.2. Alternatively one can use the Schwarz reflection principle, Rudin (1966), Theorem 11.17. We can then extend the construction for F periodically to faces $F+k_1e_1+k_2e_2$ as in Comment 2.3 (iii). This method will take care of the faces in any class $\mathfrak{F}_i$ of Comment 2.3 (iii) which contains an F which intersects L_1. If none of the faces in $\mathfrak{F}_i$ intersect L_1, then $\mathfrak{F}_i$ contains a face F in $\{0 < x(1) \leq 1\}$ and we can choose ψ symmetric with respect to $x(1) = \frac{1}{2}$ on $\overline{F} \cup \{\tilde{\overline{F}} + \xi_1\}$. This can then again be extended periodically to $\overline{F}+k_1e_1+k_2e_2$ and to $\tilde{\overline{F}}+\ell_1e_1+\ell_2e_2$, $k_i, \ell_j \in \mathbb{Z}$. The same method works if L_2: $x(2) = 0$ is an axis of symmetry. It even works if both L_1 and L_2 are axes of symmetry. In this last case ψ also has to satisfy $\psi(\tilde{x}) = \psi(x)$ as well as $\psi(\tilde{\tilde{x}}) = \widetilde{\widetilde{\psi(x)}}$, where $\tilde{\tilde{x}}$ is the reflection of x in L_2. If F intersects L_1 and L_2 we can construct such a homeomorphism ψ on F by first constructing ψ on $\overline{F} \cap \{x(1) \geq 0, x(2) \geq 0\}$ and then reflecting first in L_2 and then in L_1.

From now on we shall assume that if $(\mathcal{G},\mathcal{G}^*)$ are periodic and symmetric with respect to L_1 and/or L_2, then the same holds for $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$. In addition we can and shall assume the properties of Comments 2.2 (vii), 2.3 (i) and (v). ///

Now assume that L_i: $x(1) = a_i$, $i = 1,2$ are two vertical axes of symmetry for $\mathcal{G}_{p\ell}$ with $a_1 < a_2$. Let J be a Jordan curve in $\mathbb{R}^2$ consisting of four closed non-empty arcs B_1, A, B_2, C with disjoint interiors and occurring in this order as J is traversed in one direction. Also assume that

(2.22) For $i = 1,2$, B_i is a curve made up from edges of $\mathcal{M}_{p\ell}$, or B_i lies on L_i and J lies in the half plane $(-1)^i(x(1)-a_i) \leq 0$.

(A typical case will be that J is the perimeter of a rectangle with its left edge B_1 and right edge B_2 on an axis of symmetry.) We shall consider paths $r = (v_0, e_1, \ldots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ which satisfy the conditions (2.23)-(2.25) below.

(2.23) $$(v_1, e_2, \ldots, e_{\nu-1}, v_{\nu-1}) \subset \text{int}(J).$$

(2.24) e_1 has exactly one point in common with J. This lies in B_1 and is either v_0, or in case $B_1 \subset L_1$, it may be the midpoint of e_1.

(2.25) e_ν has exactly one point in common with J. This lies in B_2 and is either v_ν, or in case $B_2 \subset L_2$, it may be the midpoint of e_ν.

Note that if $B_1 \subset L_1$, then (2.24) implies that e_1 has to be in case (c) or case (d) of Comment 2.4 (ii). If case (d) occurs then $v_0 \in ext(J)$, because $v_1 \in int(J)$ lies to the right of L_1 and v_0 must be the reflection of v_1 in L_1. A similar comment applies to v_ν.

In several applications it will be necessary to restrict the location of r further. If S is a subset of $\mathbb{R}^2$, then $r \subset S$ will mean that all edges and vertices of r lie in S. To avoid (mild) complications we shall only consider situations with

(2.26) $$B_1 \cap B_2 \cap S = \emptyset.$$

For an r satisfying (2.23)-(2.25) we write m_0 for the unique point of e_1 on J; m_0 is either the initial point v_0 of r or the midpoint of e_1. We shall also write e_1' for the closed segment of e_1 from m_0 to v_1. We define m_ν and e_ν' similarly, and put $r' = (m_0, e_1', v_1, \dots, v_{\nu-1}, e_\nu', m_\nu)$. r' may not be an honest path on $\mathcal{G}_{p\ell}$, because e_1' and/or e_ν' may only be half an edge, while m_0 and/or m_ν may not be a vertex of $\mathcal{G}_{p\ell}$. Nevertheless, in an obvious sense, r' has no double points (see the beginning of Sect. 2.3), and the curve on $\mathcal{G}_{p\ell}$ made up from $e_1', e_2, \dots, e_{\nu-1}, e_\nu'$ is a simple arc in int(J), except for its endpoints m_0 and m_ν which lie in B_1 and B_2, respectively. Thus r' divides int(J) into two components (Newman (1951), Theorem V.11.8). On various occasions we shall use r (or r') to denote a path as well as to denote the curve made up from the edges of r (or r'). This abuse of notation is not likely to lead to confusion. For instance the components of int(J) mentioned above will be called the components of $int(J)\backslash r'$. In this notation we have

$$int(J)\backslash r = int(J)\backslash r'$$

since r differs from r' by the piece of e_1 from v_0 to m_0, excluding m_0, and the piece of e_ν from m_ν to v_ν, excluding m_ν. These pieces of e_1 and e_ν are either empty or lie to the left of L_1 and right of L_2, respectively. In either case they are contained in ext(J).

Def. 11. Let J, B_1, A, B_2 and C be as above and let r be a path on $\mathcal{G}_{p\ell}$ satisfying (2.23)-(2.25). Then $J^-(r)$ denotes the component of

$\text{int}(J)\backslash r$ which has A in its boundary, and $J^+(r)$ the component of $\text{int}(J)\backslash r$ which has C in its boundary.

To be even more explicit, $J^-(r)$ $(J^+(r))$ is the interior of the Jordan curve consisting of r' followed by the arc of J from m_ν to m_0 which contains $A(C)$.

Def. 12. If J, B_1, A, B_2 and C are as above and r_1, r_2 are two paths on $\mathcal{G}_{p\ell}$ satisfying (2.23)-(2.25) then we say that r_1 precedes r_2, and denote this by $r_1 \prec r_2$, iff $J^-(r_1) \subset J^-(r_2)$.

Proposition 2.3. Assume that (2.3)-(2.5) hold with $\mathcal{G}$ replaced by $\mathcal{M}$ and that L_i: $x(1) = a_i$, $i = 1,2$ are axes of symmetry for $\mathcal{G}_{p\ell}$, with $a_1 < a_2$. Let J be a Jordan curve consisting of four closed non-empty arcs B_1, A, B_2 and C as above satisfying (2.22). Let S be any subset of $\mathbb{R}^2$ such that (2.26) holds. Denote by $\mathcal{R} = \mathcal{R}(S,\omega)$ the collection of all occupied paths r on $\mathcal{G}_{p\ell}$ which satisfy (2.23)-(2.25) and $r \subset S$. If $\mathcal{R} \neq \emptyset$ then it has a unique element $R = R(S,\omega)$ which precedes all others. Any occupied path r on $\mathcal{G}_{p\ell}$ which satisfies (2.23)-(2.25) and $r \subset S$ also satisfies

$$r \cap \bar{J} \subset \bar{J}^+(R) \quad \text{and} \quad R \cap \bar{J} \subset \bar{J}^-(r). \tag{2.27}$$

Finally, let r_0 be a fixed path on $\mathcal{G}_{p\ell}$ satisfying (2.23)-(2.25) and $r_0 \subset S$ (no reference to its occupancy is made here). Then, whether $R = r_0$ or not depends only on the occupancies of the vertices of $\mathcal{G}_{p\ell}$ in the set

$$(\bar{J}^-(r_0) \cup V_1 \cup V_2) \cap S, \tag{2.28}$$

where $V_i = \emptyset$ if B_i is made up from edges of $\mathcal{M}_{p\ell}$, while

$$\begin{aligned} V_i = \{v:\ & v \text{ a vertex of } \mathcal{G}_{p\ell} \text{ such that its reflection } \tilde{v} \\ & \text{in } L_i \text{ belongs to } \bar{J}^-(r_0) \text{ and such that} \\ & e \cap \bar{J} \subset \bar{J}^-(r_0) \cap S \text{ for some edge } e \text{ of } \mathcal{G}_{p\ell} \text{ between} \\ & v \text{ and } \tilde{v}\}\ ,\ i = 1,2, \end{aligned}$$

in case B_i lies in L_i but is not made up from edges of $\mathcal{M}_{p\ell}$.

Another way to express the last conclusion is that for fixed r_0, the function of ω

$$I[R(\omega) \text{ exists and equals } r_0] \tag{2.29}$$

depends only on the values of $\omega(v)$ for v a vertex of $\mathcal{G}_{p\ell}$ in the set (2.28). In many applications of this proposition S will be all of $\mathbb{R}^2$, and the restriction $r \subset S$ will be vacuous in such applications. However (2.26) requires that B_1 and B_2 be disjoint for the choice $S = \mathbb{R}^2$.

2.5 Covering graphs.

Fisher (1961) and Fisher and Essam (1961) observed that a bond-percolation problem on a graph $\mathcal{G}$ is equivalent to a site-percolation problem on another graph, the socalled covering graph $\tilde{\mathcal{G}}$ of $\mathcal{G}$. We can only make this precise after the introduction of the relevant probability measures in Sect. 3.1. Here we only give the purely graph theoretical relation between $\mathcal{G}$ and $\tilde{\mathcal{G}}$.

Def. 13. Let $\mathcal{G}$ be any graph. The vertex set of the covering graph $\tilde{\mathcal{G}}$ is in a 1-1 correspondence with the edge set of $\mathcal{G}$. If $\tilde{v}_1 \neq \tilde{v}_2$ are two vertices of $\tilde{\mathcal{G}}$ corresponding to the edges e_1 and e_2 of $\mathcal{G}$ respectively, then there is one (no, two) edge of $\tilde{\mathcal{G}}$ between $\tilde{v}_1$ and $\tilde{v}_2$, if and only if e_1 and e_2 have one (no, two) endpoints in common.

Comments.

(i) Some people use the term line graph instead of covering graph.

(ii) If $\mathcal{G}$ is imbedded in $\mathbb{R}^d$ and e is an edge of $\mathcal{G}$, viewed as an arc in $\mathbb{R}^d$, then we can choose the vertex $\tilde{v}$ of $\tilde{\mathcal{G}}$ corresponding to e as a point of e. In explicit examples there is often a special choice of $\tilde{v}$ - such as the midpoint of e - and choice of edges between neighbors on $\tilde{\mathcal{G}}$ - such as line segments - which lead to a nice embedding of $\tilde{\mathcal{G}}$ (cf. Ex. 2.5 (i) below).

(iii) Let $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ be a path on $\mathcal{G}$ with possible double points and let $\tilde{v}_i$ be the vertex of $\tilde{\mathcal{G}}$ corresponding to e_i. Then there exists an edge $\tilde{e}_i$ of $\tilde{\mathcal{G}}$ between $\tilde{v}_{i-1}$ and $\tilde{v}_i$, because e_{i-1} and e_i have the endpoint v_i in common. Therefore $\tilde{r} = (\tilde{v}_1,\tilde{e}_2,\dots,\tilde{e}_\nu,\tilde{v}_\nu)$ is a path with possible double points on $\tilde{\mathcal{G}}$. If the $\tilde{v}_i$ are chosen as points on e_i, as in Comment 2.5 (i) above, then $\tilde{r}$ runs from a point of e_1 to a point at e_ν. Conversely if $\tilde{r} = (\tilde{v}_0,\tilde{e}_1,\dots,\tilde{e}_\nu,\tilde{v}_\nu)$ is a path on $\tilde{\mathcal{G}}$, with possible double points, and e_i the edge of $\mathcal{G}$ corresponding to $\tilde{v}_i$, then e_i and e_{i+1} have a common endpoint, v_i say, on $\mathcal{G}$. Then for a suitable choice of the endpoints v_0 and v_ν of e_1 and e_ν, respectively, $(v_0,e_1,\dots,e_\nu,v_\nu)$ is a path with possible double points on $\mathcal{G}$. If $\tilde{v}_i$ is a point of e_i

as in Comment 2.5 (i), then r is a path from an endpoint of e_1 to an endpoint of e_ν. This relation between paths on $\mathcal{G}$ and $\tilde{\mathcal{G}}$ is the basis for the equivalence of bond-percolation on $\mathcal{G}$ and site-percolation on $\tilde{\mathcal{G}}$ (see Prop. 3.1).

Examples.

(i) Let $\mathcal{G}$ be the hexagonal or honeycomb lattice, imbedded in $\mathbb{R}^2$ as described in ex. 2.1 (iv) (see Fig. 2.6). If we place the vertices of $\tilde{\mathcal{G}}$ at the midpoints of the edges of $\mathcal{G}$, and connect neighbors on $\tilde{\mathcal{G}}$ by straight line segments for the edges, then we see that $\tilde{\mathcal{G}}$ is the Kagomé lattice. See Fig. 2.16.

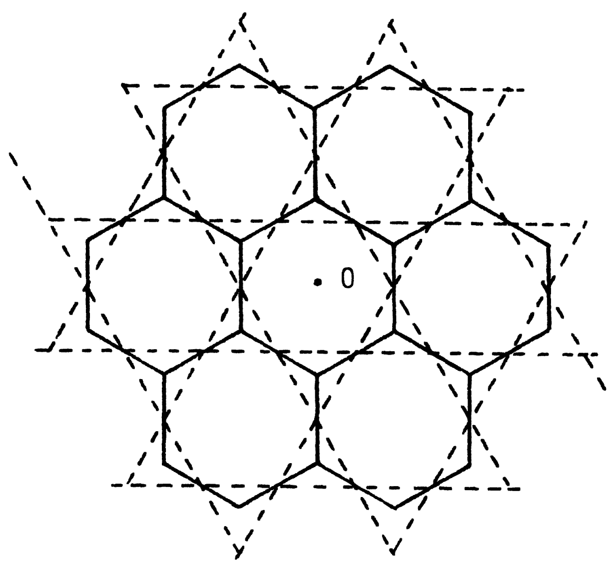

Figure 2.16 —— = hexagonal lattice, $\mathcal{G}$,
--- = Kagome lattice, $\tilde{\mathcal{G}}$.

The faces of $\tilde{\mathcal{G}}$ are regular hexagons and equilateral triangles interspersed between them.

(ii) The covering graph of the graph $\mathcal{G}_0$ in Ex. 2.1 (i) is the graph $\mathcal{G}_1$ of Ex. 2.1 (ii) (see Fig. 2.3).

2.6 Dual graphs.

Hammersley (1959), Harris (1960) and Fisher (1961) made heavy use of the socalled dual graphs in their treatment of some bond problems. The role of the dual graph is taken over by the second graph in a matching pair in our treatment, so that we shall be very brief on dual graphs. Assume that $\mathcal{G}$ is a mosaic. We then take the vertex set of the dual graph, $\mathcal{G}_d$, in 1-1 correspondence with the collection of faces of $\mathcal{G}$. In an imbedding of $\mathcal{G}$ each such face F is a Jordan domain and we place the corresponding vertex v^* of $\mathcal{G}_d$ somewhere inside F. The edge set of $\mathcal{G}_d$ is in 1-1 correspondence with the edge set of $\mathcal{G}$. Each edge of $\mathcal{G}$ lies in the perimeter of exactly two faces of $\mathcal{G}$. If e lies in the perimeter of F_1 and F_2, and v_1^* and v_2^* are the vertices of $\mathcal{G}_d$ corresponding to F_1 and F_2, respectively, then there is an edge of $\mathcal{G}_d$ between v_1 and v_2 associated to e. If the perimeters of F_1 and F_2 have ν edges in common, then there will be ν distinct edges between v_1^* and v_2^*. $\mathcal{G}_d$ has no other edges, so that v_1^* and v_2^* are neighbors if and only if they lie in adjacent faces of $\mathcal{G}$ (i.e., faces whose perimeters have an edge in common). In an imbedding of $\mathcal{G}$ we shall draw the edges of $\mathcal{G}_d$ such that an edge e^* of $\mathcal{G}_d$ intersects the unique edge e of $\mathcal{G}$ with which it is associated but no other edges of $\mathcal{G}$. One can show that if $\mathcal{G}$ is a mosaic with dual $\mathcal{G}_d$, then the covering graphs $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}_d$ of $\mathcal{G}$ and $\mathcal{G}_d$, respectively, form a matching pair. We shall not prove this, but it is easily verified for the few instances where we use dual graphs.

Examples.

(i) Take for $\mathcal{G}$ the simple quadratic lattice $\mathcal{G}_0$ of Ex. 2.1 (i). For its dual $\mathcal{G}_d$ choose a vertex at the center of each square face of $\mathcal{G}$; for the edges of $\mathcal{G}_d$ choose the line segments between the centers of adjacent square faces of $\mathcal{G}_0$ (see Fig. 2.17). $\mathcal{G}_d$ is clearly isomorphic with $\mathcal{G}_0$, in fact it is obtained by translating $\mathcal{G}_0$ by the vector $(\frac{1}{2}, \frac{1}{2})$. We say that $\mathcal{G}_0$ is self-dual.

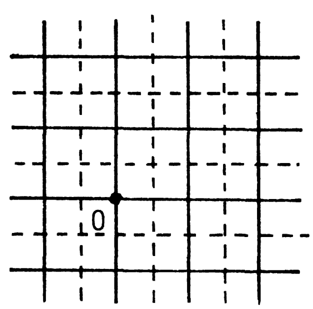

Figure 2.17 —— $= \mathcal{G} = \mathcal{G}_0$, --- $= \mathcal{G}_d$

(ii) Let $\mathcal{G}$ be the triangular lattice imbedded in $\mathbb{R}^2$ such that each face is an equilateral triangle, as in Fig. 2.4, Ex. 2.1 (iii). Choose the vertex of $\mathcal{G}_d$ corresponding to such an equilateral triangle at its center of gravity, i.e., the intersection of the bisectors of the sides of the triangle. For the edges of $\mathcal{G}_d$ take line segments along these same bisectors, and connecting the centers of gravity of adjacent triangles. $\mathcal{G}_d$ is now a copy of the hexagonal lattice of Ex. 2.1 (iv).

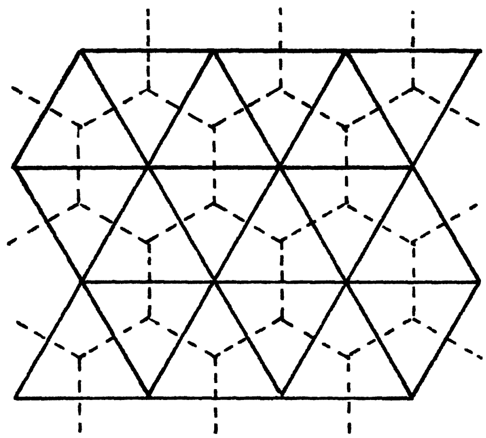

Figure 2.18 —— = $\mathcal{G}$, the triangular lattice,
- - - = $\mathcal{G}_d$, the hexagonal lattice.