11. RESISTANCE OF RANDOM ELECTRICAL NETWORKS.

11.1 Bounds for resistances of networks.

Many people have studied the electrical resistance of a network made up of random resistors. It was realized quite early that critical phenomena occur, and that there is a close relation with percolation theory, in special cases where the individual resistors can have infinite resistance (or zero resistance). We refer the reader to Kirkpatrick (1978) and Stauffer (1979) for a survey of much of this work. In these introductory paragraphs we shall assume that the reader knows what the resistance of a network is, but we shall come back to a description of resistance in Sect. 11.3.

A typical problem in which the relation with percolation is apparent is the following. Consider the graph $\mathbb{Z}^d$, with vertices the integral vectors in $\mathbb{R}^d$, and edges between two vertices v_1 and v_2 iff $|v_1-v_2| = 1$. Assume each edge of $\mathbb{Z}^d$ is a resistance of 1 ohm with probability p, and is removed with probability $q = 1-p$. As usual all edges are assumed independent of each other. Let $\mathcal{H}_n$ be the restriction of the resulting random network to the cube of size n, $B_n = [0,n]^d$. What is the behavior for large n of the resistance in $\mathcal{H}_n$ between the left and right face of B_n? More precisely let

$$(11.1)\qquad A^0 = A_n^0 = \{v = (v(1),\dots,v(d)) : v(1) = 0, 0 \le v(i) \le n, \ 2 \le i \le d\}$$

be the left face of B_n and

$$(11.2)\qquad A^1 = A_n^1 = \{v = (v(1),\dots,v(d)) : v(1) = n, 0 \le v(i) \le n, \ 2 \le i \le d\}$$

the right face. Form a new network from $\mathcal{H}_n$ by identifying as one vertex a_0 all vertices of $\mathbb{Z}^d$ in A^0, and by identifying all vertices of $\mathbb{Z}^d$ in A^1 as another vertex a_1. This means that we view all edges of $\mathcal{H}_n$ which run between the hyperplanes $x(1) = 0$ and $x(1) = 1$ as having the common endpoint a_0 in $x(1) = 0$. In "reality" one would have to connect all vertices in A^0 by wires made from some super material which has zero electrical resistance. The same has to be done for the vertices in A^1. R_n is the resistance in $\mathcal{H}_n$ between a_0 and a_1 after this identification of vertices.

For small p there will with high probability be no path at all in $\mathcal{H}_n$ connecting A_n^0 with A_n^1. Of course $R_n = \infty$ if no such path exists. Therefore

$$P_p\{R_n = \infty\} \to 1 \text{ as } n \to \infty \quad \text{for small } p.$$

On the other extreme, if $p = 1$, $\mathcal{H}_n$ becomes the restriction of $\mathbb{Z}^d$ to B_n. One easily verifies that in this situation $V(x) := x(1)/n$ is the potential at x when A_n^0 (A_n^1) is given the potential $0(1)$ by means of an external voltage source. Indeed Kirchhoff's and Ohm's laws give that $V(\cdot)$ is the unique function which satisfied

$$\sum_{\substack{y \varepsilon B_n : y \\ \text{adjacent to } x}} (V(y)-V(x)) = 0, \; x \,\varepsilon\, B_n \setminus A_n^0 \cup A_n^1$$

(i.e., $V(\cdot)$ is harmonic on $B_n \setminus A_n^0 \cup A_n^1$) and which has boundary value $0(1)$ on A_n^0 (A_n^1) (see Feynman et al (1963), Sect. I.25.4,5 and II.22.3, Nerode and Shank (1961) or Slepian (1968), Ch. 7.3; also Sect. 11.3 below). Thus, by Ohm's law the current leaving A^0 equals $(n+1)^{d-1} \frac{1}{n}$. (There are $(n+1)^{d-1}$ edges of resistance 1 ohm between A_n^0 and the hyperplane $x(1) = 1$ in B_n; the potential difference across each edge is $1/n$.) Thus, if $p = 1$, $R_n = n(n+1)^{1-d}$. It is therefore reasonable to conjecture that $n^{d-2}R_n$ converges in some sense to a finite and non-zero (random) limit as $n \to \infty$, at least when p is large enough. We do not know how to prove such a result, but the results in this chapter establish that n^{2-d} gives the correct order of magnitude of R_n when $p > \frac{1}{2}$. For $d = 2$ we obtain much more precise information on R_n for all p.

Of course removing an edge e is equivalent to giving e an infinite resistance. A dual problem arises when each edge of $\mathbb{Z}^d$ is a resistor of 1 ohm (has zero resistance) with probability $p(q = 1-p)$. The resistance R_n between A_n^0 and A_n^1 of the restriction to B_n of $\mathbb{Z}^d$ will now be zero as soon as there exists a single path in B_n containing only edges of zero resistance and connecting A_n^0 with A_n^1. The probability of $\{R_n = 0\}$ will therefore tend to one as $n \to \infty$ for large enough p, but for small p $n^{d-2}R_n$ should be bounded away from zero.

In the theorems below we shall combine both situations. In fact we shall allow for an arbitrary distribution of the resistances of the individual edges. We restrict ourselves to the graph $\mathbb{Z}^d$, and for most of the results even to $\mathbb{Z}^2$. It is clear, however, that a good part of the method of proof used for $\mathbb{Z}^2$ will work when $\mathbb{Z}^2$ is replaced by another graph $\mathcal{G}$ imbedded in $\mathbb{R}^2$ which is one of a matching pair $(\mathcal{G},\mathcal{G}^*)$ for which $p_T(\mathcal{G}) = p_S(\mathcal{G}) = 1-p_T(\mathcal{G}^*)$.

Before formulating our results we point out that continuum analogues of the resistance problem have been studied as well. For instance Papanicolaou and Varadhan (1979) and Golden and Papanicolaou (1982) (see especially Appendix) assume that the conductivity of a certain material is a random process, indexed by position in $\mathbb{R}^d$ (rather than time). This process is assumed stationary. Under suitable assumptions the asymptotic behavior of the conductivity of B_n between A_n^0 and A_n^1 for the random medium is the same as that of a certain deterministic "effective" medium. Golden and Papanicolaou (1982) give bounds for the conductivity of the effective medium. A related sequence of bounds for the conductivity in a composite medium can be found in Milton (1981). However, these bounds seem to apply only for a material of two components, both of which have a finite non-zero conductivity.

We turn to the precise formulation of our theorems. It turns out that in the first mentioned problem a good way to estimate R_n is to find a lower bound for the number of disjoint paths in $\mathcal{H}_n$ from A_n^0 to A_n^1. In other words, we try to find many disjoint conducting paths (i.e., paths each of whose edges has finite resistance) in B_n from the left to the right face. This part of the analysis is pure percolation theory. For a closer match with the previous chapters we treat this part as a site percolation problem. Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^d$. By definition of $p_S = p_S(\mathcal{G})$ (see (3.65)) the probability under P_p that there exists any occupied path on $\mathcal{G}$ in B_n from A_n^0 to A_n^1 tends to zero (as $n \to \infty$) whenever $p < p_S$. For many of the graphs in $\mathbb{R}^2$ which we considered the same probability tends to one when $p > p_S$. We now define a new critical probability as the dividing point where lots of disjoint occupied paths in B_n from A_n^0 to A_n^1 begin to appear. Specifically, we want of the order of n such paths. With P_p the one-parameter probability measure defined by (3.22) and (3.61) and i-crossings as in Def. 3.1 we define

(11.3) $\hat{p}_R = \hat{p}_R(\mathcal{G}) = \inf\{p: \exists\ C(p) > 0$ such that
$P_p\{\exists\ C(p)n$ disjoint occupied 1-crossings of $[0,n]^d$
for all large $n\} = 1\}$.

Clearly $\hat{p}_R \geq p_S$. It is also not hard to show that in general (cf. (1.16))

$$p_R \leq \hat{p}_R \ .$$

This is an easy consequence of the fact that the harmonic mean is less than or equal to the arithmetic mean. The proof of this relation between p_R and $\hat{p}_R$ is implicit in the proof of Theorem 11.2 (see the lines preceding (11.81). We shall not make this proof any more explicit. Instead we concentrate on the much harder and more crucial relation

$$\hat{p}_R = p_S \ ,$$

which holds for many graphs in $\mathbb{R}^2$. Since we are also interested in an estimate for $C(p)$ in (11.3) in terms of powers of $(p-p_H(\mathcal{G}))$ we want to appeal to the results of Ch. 8. We therefore restrict ourselves to proving $\hat{p}_R = p_S$ only for the graphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$ introduced in Ex. 2.1(i), 2.1(ii), 2.2(i) and 2.2(ii). (See, however, Remark (i) below.)

<u>Theorem 11.1</u>. <u>Let</u> $\mathcal{G}$ <u>be one of the graphs</u> $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ <u>or</u> $\mathcal{G}_1^*$ <u>and let</u> P_p <u>be the one-parameter probability measure on the occupancy configurations of</u> $\mathcal{G}$ <u>of the form (3.22) and specified by (3.61). Then for some universal constants</u> $0 < C_i, \delta_i < \infty$ <u>one has</u>

(11.4) $P_p\{\exists$ at least $C_1(p-p_H(\mathcal{G}))^{\delta_1}n$ disjoint occupied
horizontal crossings on $\mathcal{G}$ of $[0,m]\times[0,n]\}$
$\geq 1-C_2(m+1)\exp\{-C_3(p-p_H(\mathcal{G}))^{\delta_2}n\}$,

<u>whenever</u> $p \geq p_H(\mathcal{G})$.

<u>Remarks.</u>

(i) The proof given below can be used to show that for any $\mathcal{G}$ which is one of a matching pair of periodic graphs in $\mathbb{R}^2$ there exist constants $0 < C_i = C_i(p,\mathcal{G}) < \infty$ for which

(11.5) $P_p\{\exists$ at least C_1n occupied horizontal crossings on
$\mathcal{G}$ of $[0,m]\times[0,n]$, no pair of which has a vertex in
common$\} \geq 1-C_2m\ \exp -C_3n$,

whenever $p > 1-p_T(\mathcal{G}^*)$.

(ii) The proof below for Theorem 11.1 is largely taken from Grimmett and Kesten (1982). The estimate in the latter paper does, however, differ slightly from (11.4). It replaces $C_1(p-p_H(\mathcal{G}))^{\delta_1}n$ inside the braces in (11.4) by $(\mu-\varepsilon)n$, where μ is the time constant of a certain first-passage percolation problem (see Smythe and Wierman (1978) for such problems) and then shows that on $\mathbb{Z}^2$ the number of edge disjoint occupied crossings of $[0,m]\times[0,n]$ is actually of the order μn for $n^{-1}\log m$ small. The fact that one can give a lower bound of the form $C_1(p-p_H(\mathcal{G}))^{\delta_1}$ for the time constant μ of the first passage percolation under consideration was pointed out to the author by J. T. Cox (private communication). It is this observation which leads to the $C_1(p-p_H(\mathcal{G}))^{\delta_1}n$ in the left hand side of (11.4). Grimmett and Kesten (1982) do not pursue the dependence on p of the various constants, but instead are interested in the exponential bound (11.5) with C_1 as large as possible (all the way up to μ). ///

We return to the resistance problem on $\mathbb{Z}^d$. We shall assume that each edge has a resistance $R(e)$ with all $R(e)$, e an edge of $\mathbb{Z}^d$, independent random variables, all with the same distribution given as follows:

(11.6) $$P\{R(e) = 0\} = p(0),$$

(11.7) $$P\{R(e) = \infty\} = p(\infty),$$

and for any Borel set $B \subset (0,\infty)$

(11.8) $$P\{R(e) \in B\} = \int_B dF(x)$$

for some measure F on $(0,\infty)$ with total mass $1-p(0)-p(\infty)$.

In the next two theorems $\theta_d(p)$ will denote the percolation probability for bond-percolation on $\mathbb{Z}^d$ under the measure P_p according to which all edges are independently open or passable (blocked) with probability $p(q = 1-p)$. Thus, for any edge e of $\mathbb{Z}^d$

$$\theta_d(p) = P_p\{e \text{ belongs to an infinite open cluster}\}.$$

Also $p_{S,d}$ will be the critical probability of (3.65) for bond-percolation on $\mathbb{Z}^d$. In Theorem 11.2 we take $d = 2$ so that in (11.11) $\theta_2(p)$ also equals the percolation probability for site-percolation on $\mathcal{G}_1$, i.e., $\theta_2(p) = P_p\{v$ belongs to an infinite occupied cluster on $\mathcal{G}_1\}$ for v any vertex of $\mathcal{G}_1$ and P_p given by (3.22) and (3.61) with $\mathcal{V}$ the vertex set of $\mathcal{G}_1$ (compare Ch. 3.1; $\mathcal{G}_1$ is introduced in Ex. 2.1(ii)).

Theorem 11.2. Assume the edges of $\mathbb{Z}^2$ have independent resistances with distribution given by (11.6)-(11.8). Let R_n be the resistance between A_n^0 and A_n^1 of the network in B_n. Then

$$P\{R_n = 0 \text{ eventually}\} = 1 \quad \text{if} \quad p(0) > \frac{1}{2}, \tag{11.9}$$

$$P\{R_n = \infty \text{ eventually}\} = 1 \quad \text{if} \quad p(\infty) > \frac{1}{2}. \tag{11.10}$$

Moreover there exist constants $0 < C_i, \delta_i < \infty$ such that if $p(0) < \frac{1}{2}$ and $p(\infty) < \frac{1}{2}$, then

$$P\{C_4 \frac{(\frac{1}{2} - p(0))^{2\delta_1}}{\theta_2(1-p(\infty))} \{\int_{(0,\infty)} \frac{1}{x} dF(x)\}^{-1} \le \liminf_{n\to\infty} R_n \tag{11.11}$$

$$\le \limsup_{n\to\infty} R_n \le C_5 \frac{\theta_2(1-p(0))}{(\frac{1}{2} - p(\infty))^{2\delta_1}} \int_{(0,\infty)} x\, dF(x)\} = 1,$$

(δ_1 is the same as in (11.4)).

Corollary 11.1. Let the set up be the same as in Theorem 11.2. Then there exist constants $0 < C_i, \delta_i < \infty$ such that for $p(0) = 0$, $p(\infty) < \frac{1}{2}$ one has

$$P\{C_6(\frac{1}{2} - p(\infty))^{-\delta_3}\{\int_{(0,\infty)} \frac{1}{x} dF(x)\}^{-1} \le \liminf R_n \tag{11.12}$$

$$\le \limsup R_n \le C_5(\frac{1}{2} - p(\infty))^{-2\delta_1} \int_{(0,\infty)} x\, dF(x)\} = 1.$$

If, on the other hand, $p(0) < \frac{1}{2}$, $p(\infty) = 0$, then

$$P\{C_4(\frac{1}{2} - p(0))^{2\delta_1}\{\int_{(0,\infty)} \frac{1}{x} dF(x)\}^{-1} \le \liminf R_n \tag{11.13}$$

$$\le \limsup R_n \le C_7(\frac{1}{2} - p(0))^{\delta_3}\int_{(0,\infty)} x\, dF(x)\} = 1.$$

Remarks.

(iii) If $\int_{(0,\infty)} \frac{1}{x} dF(x) = \infty$ then $\{\int_{(0,\infty)} \frac{1}{x} dF(x)\}^{-1}$ is to be interpreted as zero, and the lower bounds for R_n in (11.11)-(11.13) become vacuous. Similarly the upper bounds become vacuous if $\int_{(0,\infty)} x\, dF(x) = \infty$. Nevertheless it is possible to use Theorem 11.2 to obtain non-trivial bounds for R_n in such cases by truncation. For example, assume that $\int x\, dF(x) = \infty$ and $p(0) < \frac{1}{2}$, $p(\infty) < \frac{1}{2}$. Define m as the unique finite number for which

$$F((m,\infty)) \le \tfrac{1}{2}(\tfrac{1}{2} - p(\infty)) \le F([m,\infty)).$$

Now take for each edge e

$$R'(e) = R(e) \quad \text{if } R(e) < m,$$

$$R'(e) = \infty \quad \text{if } R(e) > m \ \text{(including } R(e) = \infty) .$$

If $R(e) = m$, then randomize again for $R'(e)$ and take

$$R'(e) = m \ \text{ with probability } \ F([m,\infty)) - \tfrac{1}{2}(\tfrac{1}{2} - p(\infty))$$

and

$$R'(e) = \infty \ \text{ with probability } \ \tfrac{1}{2}(\tfrac{1}{2} - p(\infty)) - F((m,\infty)).$$

Again the randomizations for $R'(e)$ when $R(e) = m$ are done independently for all edges. Then $R'(e) \ge R(e)$ for all e, and if R_n' denotes the resistance between A_n^0 and A_n^1 in B_n when we use the $R'(e)$ instead of the $R(e)$ then (see Lemma 11.4 below)

$$R_n' \ge R_n .$$

Since

$$P\{R'(e) = \infty\} = p(\infty) + \tfrac{1}{2}(\tfrac{1}{2} - p(\infty)),$$

we obtain from (11.11) applied to R_n'

$$\text{(11.14)} \qquad P\{\limsup R_n \le \limsup R_n' \le C_5 2^{2\delta_1} \frac{\theta_2(1-p(0))}{(\frac{1}{2} - p(\infty))^{2\delta_1}} \int_{(0,m]} x\, dF(x)\} = 1$$

whenever $m > 0$. In a similar way one can truncate $R(e)$ near zero to obtain a nonzero lower bound for $\liminf R_n$ when $p(0) < \frac{1}{2}$ and $\int \frac{1}{x}\, dF(x) = \infty$.

(iv) Theorem 11.2 as stated gives no information when $p(0) = \frac{1}{2}$ or $p(\infty) = \frac{1}{2}$. Actually, from (11.11) and simple monotonicity arguments one obtains

(11.15) $P\{\lim R_n = \infty\} = 1$ if $p(0) < \frac{1}{2} = p(\infty)$

and

(11.16) $P\{\lim R_n = 0\} = 1$ if $p(0) = \frac{1}{2} > p(\infty)$.

For example, to obtain (11.15) one merely has to randomize $R(e)$ when $R(e) = \infty$ and to take

$$R''(e) = \begin{cases} 1 & \text{with probability } \varepsilon \\ \infty & \text{with probability } 1-\varepsilon , \end{cases}$$

but to take $R''(e) = R(e)$ when $R(e) < \infty$ (11.15) is then obtained by applying (11.11) to the $R''(e)$ instead of $R(e)$ and taking the limit as $\varepsilon \downarrow 0$.

Finally, when $p(0) = p(\infty) = \frac{1}{2}$, F = zero measure, then

(11.17) $$\lim_{n\to\infty} P\{R_n = 0\} = \lim_{n\to\infty} P\{R_n = \infty\} = \frac{1}{2} .$$

We shall not prove (11.17), but merely note that if each $R(e) = 0$ or ∞ then also $R_n = 0$ or ∞, and $R_n = 0$ if and only if there is a path in B_n from A_n^0 to A_n^1 all of whose edges have zero resistance. The probability of this event is precisely the sponge-crossing probability $S_{1/2}(n+1,n+1)$ of Seymour and Welsh (1978) and Seymour and Welsh (1978, pp.233, 234) already show $S_{1/2}(n+1,n+1) \geq S_{1/2}(n,n+1) = \frac{1}{2}$.

(v) When $p(0) = p(\infty) = 0$ percolation theory does not really enter. One can then trivially estimate R_n from above by the resistance of the network consisting of the (n+1) parallel (disjoint) paths $\{k\} \times [0,n]$, $k = 0,\ldots,n+1$. A very much simplified version of the proofs of Theorems 11.2 and 11.3 then yields

(11.18) $$\left\{\int_{(0,\infty)} \frac{1}{x}\, dF(x)\right\}^{-1} \leq \liminf R_n \leq \limsup R_n \leq \int_{(0,\infty)} x\,dF(x).$$

This bound has apparently been known for a long time (see Milton (1981) and its references).

(vi) It seems very likely that $\lim n^{d-2} R_n$ exists in some sense. Golden and Papanicolaou (1982), Appendix, show that R_n converges in $L^2(P)$ when $d = 2$, $p(0) = p(\infty) = 0$ and F concentrated on an interval $[a,b]$, $0 < a < b < \infty$. Their proof actually deals with the continuum analogue but appears to apply as well in our set up. Straley (1977) uses duality arguments to discuss the case $d = 2$, $p(0) = p(\infty) = 0$, $P\{R(e) = a\} = P\{R(e) = b\} = \frac{1}{2}$ for some $0 < a < b < \infty$. These arguments show that in whatever sense R_n has a limit, the value of the limit should be $(a\,b)^{1/2}$. In particular by the above result of Golden and Papanicolaou $E\{R_n-(ab)^{1/2}\}^2 \to 0$ as $n \to \infty$. ///

For $d > 2$ our results are quite incomplete.

Theorem 11.3. Assume the edges of $\mathbb{Z}^d$ have independent resistances with distribution given by (11.6)-(11.8). Let R_n be the resistance between A_n^0 and A_n^1 of the network in B_n. Then

$$P\{R_n = 0 \text{ eventually}\} = 1 \quad \text{if} \quad p(0) \geq \frac{1}{2} \tag{11.19}$$

and

$$P\{R_n = \infty \text{ eventually}\} = 1 \quad \text{if} \quad 1-p(\infty) < p_{S,d} . \tag{11.20}$$

Moreover, there exist constants $0 < C_i, \delta_i < \infty$ such that

$$P\{\liminf n^{d-2}R_n \geq \frac{1-p(\infty)}{\theta_d(1-p(\infty))} \{\int_{(0,\infty)} \frac{1}{x}\, dF(x)\}^{-1}\} = 1 \tag{11.21}$$

$$\text{if} \quad p(0) = 0,$$

and

$$P\{\limsup n^{d-2}R_n \leq \frac{C_9}{(\frac{1}{2} - p(\infty))^{2\delta_1}} \int_{(0,\infty)} x\, dF(x)\} = 1 \tag{11.22}$$

$$\text{if} \quad p(0) < \frac{1}{2} \text{ and } p(\infty) < \frac{1}{2} .$$

(δ_1 is the same as in (11.4).)

11.2 Proof of Theorem 11.1.

We shall only prove a weakened version of (11.4). This will suffice for Theorems 11.2 and 11.3. Instead of obtaining $C_1(p-p_H(\mathcal{G}))^{\delta_1}n$ disjoint occupied crossings of $[0,m]\times[0,n]$ as desired in (11.4) we only obtain this many crossings no pair of which has a vertex in common. Thus the occupied crossings are only vertex-disjoint. For a planar graph such as $\mathcal{G}_0$ this means that the crossings are actually disjoint, but not for a non-planar graph such as $\mathcal{G}_1$. To obtain disjoint crossings one should carry out the argument below (with a number of complicating modifications) on $\mathcal{G}_{p\ell}$.

Now for the proof of the weakened version of (11.4). We restrict ourselves to the case $\mathcal{G} = \mathcal{G}_1$, the other cases being quite similar. We take for A_1 the zig zag curve strictly to the left of $\{0\}\times[0,n]$, starting at $(-1,\frac{1}{2})$, going to $(-\frac{3}{2},1)$, then to $(-1,\frac{3}{2})$ and extended periodically, with final point $(-1,n-\frac{1}{2})$. Similarly A_3 is a zig-zag curve strictly to the right of $\{m\}\times[0,n]$, from $(m+1,\frac{1}{2})$ to $(m+1,n-\frac{1}{2})$ (see Fig. 11.1). Also A_2 is a zig-zag curve from $(-1,\frac{1}{2})$ to $(m+1,\frac{1}{2})$ lying strictly above $[0,m]\times\{0\}$ and obtained by periodic repetition of the segments from $(-1,\frac{1}{2})$ to $(-\frac{1}{2},1)$ to $(0,\frac{1}{2})$ etc. A_4 is a similar path strictly below $[0,m]\times\{n\}$ from $(-1,n-\frac{1}{2})$ to $(m+1,n-\frac{1}{2})$. The composition of A_1-A_4 is a Jordan curve on $\mathcal{M}$, where $\mathcal{M}$ is the mosaic on which $\mathcal{G}_1$ is based ($\mathcal{M}$ is $\mathbb{Z}^2$ rotated over 45° and

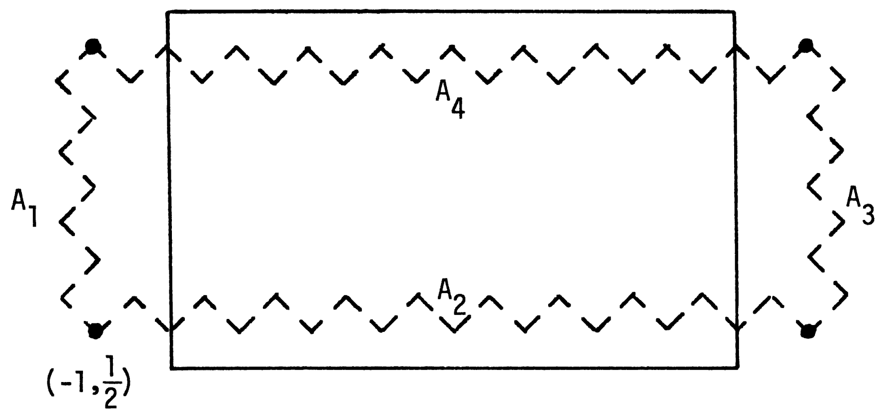

Figure 11.1 The solid rectangle is $[0,m]\times[0,n]$
The dashed curve is J.

translated by $(\frac{1}{2},0)$. $\mathcal{G}_1$ is imbedded as in Ex. 2.1(ii); see Fig.2.3). Any path on $\mathcal{G}_1$ in $\bar{J} = J \cup \text{int}(J)$ from a vertex on A_1 to a vertex on A_3, and with all its sites in $\bar{J}\setminus A_1 \cup A_3$ occupied, contains an occupied crossing on $\mathcal{G}_1$ of $[0,m]\times[0,n]$. Moreover, if two such paths in $\bar{J}$ have no vertex in common in $\bar{J}\setminus A_1 \cup A_3$, then they have no vertex in common in $[0,m]\times[0,n]$. It therefore suffices to find a lower bound on the maximal number of paths on $\mathcal{G}_1$ in $\bar{J}$ from A_1 to A_3 with only occupied vertices in $\bar{J}\setminus A_1 \cup A_3$, and such that any pair of these paths has no common vertices in $\bar{J}\setminus A_1 \cup A_3$.

To find the desired lower bound fix an occupancy configuration ω and form the graph $\mathcal{G}_J(\omega)$, which is basically the restriction of the occupied part of $\mathcal{G}$ to $\bar{J}$, with the vertices on A_1 identified as one vertex $\hat{1}$, and similarly for the vertices on A_3. More precisely, the vertex set of $\mathcal{G}_J(\omega)$ consists of the collection of vertices v of $\mathcal{G}$ in $\bar{J}\setminus A_1 \cup A_3$ which are occupied in ω plus two vertices which we denote by $\hat{1}$ and $\hat{3}$. There is an edge between two vertices v' and v'' of $\mathcal{G}_J(\omega)$ (other than $\hat{1}$ and $\hat{3}$) iff there is an edge between them in $\mathcal{G}$. There is no edge between $\hat{1}$ and $\hat{3}$, and there is an edge between v and $\hat{1}(\hat{3})$ if there is an edge in $\mathcal{G}$ between v and some vertex on $A_1(A_3)$. We shall now apply Menger's Theorem - which is a version of the max flow-min cut theorem, see Bollobás (1979), Theorem III.5(i) - to $\mathcal{G}_J(\omega)$. In the terminology of Bollobás, the number of independent paths from $\hat{1}$ to $\hat{3}$ in $\mathcal{G}_J(\omega)$ equals the maximal number of paths on $\mathcal{G}_J(\omega)$ from A_1 to A_3 which are pairwise vertex-disjoint on $\bar{J}\setminus A_1 \cup A_3$. In turn, this is precisely the maximal number of paths on $\mathcal{G}_1$ in $\bar{J}$ from A_1 to A_3 all of whose vertices in $\bar{J}\setminus A_1 \cup A_3$ are occupied and which are pairwise vertex-disjoint on $\bar{J}\setminus A_1 \cup A_3$. By Menger's Theorem this number equals the minimum of the cardinalities of sets of vertices of $\mathcal{G}_J(\omega)$ which separate $\hat{1}$ from $\hat{3}$. (A set V of vertices in $\mathcal{G}_J(\omega)\setminus\{\hat{1},\hat{3}\}$ separates $\hat{1}$ from $\hat{3}$ if after removal of V there no longer exists a path from $\hat{1}$ to $\hat{3}$ on $\mathcal{G}_J(\omega)$). Let us denote by L this minimum cardinality of separating sets. Now consider the collection of all paths $r^* = (v_0^*,e_1^*,\dots,e_\nu^*,v_\nu^*)$ on $\mathcal{G}^*$ which satisfy

$$r^* \subset \bar{J}\setminus A_1 \cup A_3 \tag{11.23}$$

and

$$v_0^* \text{ lies on } \mathring{A}_2,\ v_\nu^* \text{ lies on } \mathring{A}_4\ . \tag{11.24}$$

Denote by M the minimal number of occupied vertices on any such path. We shall use Prop. 2.2 to show that

$$L = M.$$

First pick a path r^* which satisfies (11.23) and (11.24) and which contains only M occupied vertices. If we modify ω by making these M vertices vacant, then r^* becomes vacant and then by Prop. 2.2 there is no longer a path on $\mathcal{G}$ in $\bar{J}$ from A_1 to A_3 with all its vertices in $\bar{J}\setminus A_1 \cup A_3$ occupied. Equivalently, removal of the M occupied vertices on r^* separates A_1 from A_3, so that the occupied vertices on r^* form a set which separates $\hat{1}$ from $\hat{3}$ in $\mathcal{G}_J(\omega)$, whence $L \le M$. Conversely, let V be a set of L vertices which separates $\hat{1}$ from $\hat{3}$ on $\mathcal{G}_J(\omega)$. View V also as a set of vertices of $\mathcal{G}_1$ and let $\tilde{\omega}$ be the occupancy configuration obtained by making the vertices in V vacant. Making the vertices of V vacant amounts to removing them from $\mathcal{G}_J(\omega)$, i.e., $\mathcal{G}_J(\tilde{\omega})$ does not contain any vertex from V. Since V was a separating set $\hat{1}$ and $\hat{3}$ are not connected by a path on $\mathcal{G}_J(\tilde{\omega})$. Therefore, in the configuration $\tilde{\omega}$, A_1 and A_3 are not connected by a path on $\mathcal{G}_1$ in $\bar{J}$ with all its vertices on $\bar{J}\setminus A_1 \cup A_3$ occupied. By Prop. 2.2 this means that there exists a path r^* on $\mathcal{G}^*$ which satisfies (11.23) and (11.24) and which is vacant in the configuration $\tilde{\omega}$. Since $\tilde{\omega}$ differs from ω only on the vertices of V, it follows that r^* has at most $\#V = L$ occupied vertices in the configuration ω. Thus, $M \le L$ and $M = L$ as claimed.

So far we have shown that

(11.25) number of vertex-disjoint occupied horizontal crossings on $\mathcal{G}$ of $[0,m]\times[0,n]$ in the configuration
$\ge$ maximal number of paths on $\mathcal{G}_J(\omega)$ from $\hat{1}$ to $\hat{3}$, such that no pair has a vertex in common in $\bar{J}\setminus A_1 \cup A_3$
$= L = M$.

Thus, the left hand side of (11.25) can be less than $C_1(p-p_H(\mathcal{G}_1))^{\delta_1}$ $= C_1(p-\frac{1}{2})^{\delta_1}n$ only if also $M < C_1(p-\frac{1}{2})^{\delta_1}n$. Now note that A_4 lies on or above the horizontal line $\mathbb{R}\times\{n-1\}$. Therefore, $M_\circ < C_1(p-\frac{1}{2})^{\delta_1}n$ can happen only if one of the $(2m+3)$ vertices v^* of $\mathring{A}_2$ is connected to the horizontal line $\mathbb{R}\times\{n-1\}$ by a path on $\mathcal{G}_1^*$ containing fewer than $C_1(p-\frac{1}{2})^{\delta_1}n$ occupied vertices. Consequently

(11.26) $P_p\{$maximal number of vertex-disjoint occupied horizontal crossings on $\mathcal{G}_1$ of $[0,m]\times[0,n]$ is less than $C_1(p-\frac{1}{2})^{\delta_1}n\}$
$\le (2m+3) \max_{v^*(2)\le 1} P_p\{\exists$ path on $\mathcal{G}_1^*$ from v^* to the half plane $\mathbb{R}\times[n-1,\infty)$ which contains fewer than $C_1(p-\frac{1}{2})^{\delta_1}n$ occupied vertices$\}$.

To estimate the right hand side of (11.26) we use a slightly strengthened verions of (5.48) or Lemma 1 of Kesten (1980b). For a closer agreement with the notation of Lemma 5.4 we interchange the role of $\mathcal{G}$ and $\mathcal{G}^*$ as well as the role of "occupied" and "vacant" and the role of the first and second coordinate. Theorem 11.1 is then immediate from (11.26) and the Proposition below. (Recall that $p_H(\mathcal{G}^*) = 1-p_H(\mathcal{G})$ for $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$ by Applications 3.4(iv) and (ii).) □

Proposition 11.1. Let $\mathcal{G}$ be one of the graphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ or $\mathcal{G}_1^*$ and let P_p be the one-parameter probability measure on the occupancy configurations of $\mathcal{G}$ of the form (3.22) and specified by (3.61). Then for some universal constants $0 < C_i, \delta_i < \infty$ one has for any vertex v of $\mathcal{G}$ with $v(1) \leq 0$ and $p \leq p_H = p_H(\mathcal{G})$

$$(11.27)\qquad P_p\{\exists \text{ path from } v \text{ to } [n,\infty)\times\mathbb{R} \text{ on } \mathcal{G} \text{ which contains fewer than } C_6(p_H-p)^{\delta_1}n \text{ vacant vertices}\} \leq 10\exp-C_7(p_H-p)^{\delta_2}n.$$

Proof: As in Lemma 5.4 we set for any vertex u of $\mathcal{G}$ and integer M

$$S_0 = S_0(v,M) = \{w \text{ a vertex of } \mathcal{G}: |w(j)-v(j)| \leq M,\ j = 1,2\},$$

$$S_1 = S_0 \cup \partial S_0 = \{w \text{ a vertex of } \mathcal{G}: w \in S_0 \text{ or } w \text{ adjacent to a vertex in } S_0\}.$$

Instead of $A(u,m)$ in (5.47) we now define for positive integers n and k the event

$$A(v,n,k) = \{\exists \text{ a path on } \mathcal{G} \text{ from a neighbor of } v \text{ to a } w \text{ with } w(1) \geq n \text{ which contains at most } k \text{ vacant vertices}\}.$$

We repeat the definition of g from Lemma 5.4.

$$g(v,w,M) = P_p\{\exists \text{ occupied path } (w_0,e_1,\dots,e_\rho,w_\rho) \text{ on } \mathcal{G} \text{ with } w_0\mathcal{G}v,\ w_\rho \notin S_0(v,M) \text{ one of the } w_i \text{ equal to } w\}.$$

The principal estimate is the following strengthened version of (5.48).[1)]For $v(1) < n-M$ and $k \geq 0$,

---

1) One could also use the argument of Lemma 1 in Kesten (1980b), which avoids the use of the random set R. However, the present argument is a closer parallel to Ch. 5 and needs essentially no new steps.

$$(11.28)\qquad P_p\{A(v,n,k)\} \le \sum_{w\varepsilon S_1(v,M)} g(v,w,M)P_p\{A(w,n,k)\} + \sum_{w\varepsilon S_1(v,M)} P_p\{A(w,n,k-1)\} .$$

To prove (11.28) let E be the event

$$E = \bigcup_{w\varepsilon S_1(v,M)} A(w,n,k-1).$$

Clearly the second term in the right hand side of (11.28) is an upper bound for $P_p\{E\}$ so that we only have to estimate $P_p\{A(v,n,k)\setminus E\}$. Assume then that $A(v,n,k)\setminus E$ occurs and that $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ is a path on $\mathcal{G}$ with v_0 adjacent to v, $v_\nu(1) \ge n$ and such that r contains at most k vacant vertices. Since $v_\nu(1) \ge n > v(1)+M$, v_ν must lie outside $S_0(v,M)$ and there exists a smallest index a with $v_a \notin S_0(v,M)$. Since $v_{a-1} \varepsilon S_0(v,M)$ we still have $v_a \varepsilon \partial S_0 \subset S_1$. Since E does not occur $(v_{a+1},e_{a+2},\dots,e_\nu,v_\nu)$ must contain more than $(k-1)$ vacant vertices (otherwise $A(v_a,n,k-1)$ would occur). But then $(v_0,e_1,\dots,e_{a-1},v_a)$ cannot contain any vacant vertex, because r contains at most k vacant vertices. Consequently with R defined as in Lemma 5.4 $v_a \varepsilon R$. As in Lemma 5.4 let $b \ge a$ be the last index with $v_b \varepsilon R$. Then the path $(v_{b+1},e_{b+2},\dots,e_\nu,v_\nu)$ has all its vertices outside R, its initial point, v_{b+1}, is adjacent to $v_b \varepsilon R$ and its final point v_ν satisfies $v_\nu(1) \ge n$. Moreover this path is a subpath of r and therefore contains at most k vacant sites so that $A(v_b,n,k)$ occurs. Thus, as in (5.49)

$P_p\{A(v,n,k)\setminus E\} \le \sum_{w\varepsilon S_1} P_p\{w \varepsilon R$ and $\exists$ a path $(w_0,f_1,\dots,f_\rho,w_\rho)$ on $\mathcal{G}$ with $w_0 \mathcal{G} w$, $w_\rho(1) \ge n$, $w_i \notin R$ for $0 \le i \le \rho$ and at most k of the w_i, $0 \le i \le \rho$, are vacant$\}$.

One can now copy the argument following (5.48) practically word for word to obtain

$$(11.29)\qquad P_p\{A(v,n,k)\setminus E\} \le \sum_{w\varepsilon S_1} g(v,w,M)P_p\{A(w,n,k)\};$$

one only has to replace "occupied path" in the definition (5.50) of J by "path with at most k vacant vertices". The right hand side of (11.29) is just the first sum in the right hand side of (11.28) so that (11.28) follows.

In order to exploit (11.28) we must now choose M such that (5.51) holds. This time we must also keep track of the dependence of M on p. But, for every v, w, M

$g(v,w,M) \leq P_p\{$some neighbor of v is connected by an occupied path on $\mathcal{G}$ to $\partial S_0(v,M)\} \leq P_p\{\exists$ on occupied horizontal crossing of $[v(1)-M,v(1)-1]\times[v(2)-M,v(2)+M]$ or of $[v(1)+1,v(1)+M]\times[v(2)-M,v(2)+M]\} + P_p\{\exists$ an occupied vertical crossing of $[v(1)-M,v(1)+M]\times[v(2)-M,v(2)-1]$ or of $[v(1)-M,v(1)+M]\times[v(2)+1,v(2)+M]\}$,

since any path from a neighbor of v to $\partial S_0(v,M)$ must cross either the left or right "half", or the bottom or top "half" of $S_0(v,M)$. With the notation of (5.5) we therefore conclude from Comment 3.3(v) and Lemma 8.3 that

$$(11.30)\qquad g(v,w,M) \leq 2\tau((M-1,M-1);1,p)+2\tau((M-1,M-1);2,p) \leq 4 \exp -C_{13}(p_H-p)(M-1)^{\alpha_1} .$$

Since $S_1(v,M)$ contains at most $2(2M+3)^2$ vertices of $\mathcal{G}_1$ we have

$$(11.31)\qquad \sum_{w\varepsilon S_1(v,M)} g(v,w,M) \leq 8(2M+3)^2 \exp -C_{13}(p_H-p)(M-1)^{\alpha_1} \leq \frac{3}{4}$$

for

$$(11.32)\qquad M = C_8(p_H-p)^{-2/\alpha_1}$$

for some suitable C_8, which depends on C_{13} only (C_{13} is as in Lemma 8.3).

From here on the proof is identical with that of Prop. 1 of Kesten (1980b). We choose M such that (11.32) and hence (11.31) hold. We rewrite (11.28) as

$$(11.33)\qquad P_p\{A(v,n,k)\} \leq \sum_{w_1,y_1} h(v,w_1,y_1)P_p\{A(v+w_1,n,k-y_1)\},$$

where w_1 runs over those points in the square $[-M-1,M+1]\times[-M-1,M+1]$ for which $v+w_1 \varepsilon S_1(v,M)$ and y_1 takes the values 0 or 1. Finally

$$h(v,w_1,y_1) = \begin{cases} 1 & \text{if } y_1 = 1 , \\ g(v,v+w_1,M) & \text{if } y_1 = 0 . \end{cases}$$

Next we observe that

$$P_p\{A(w,n,-1)\} = 0 \quad \text{if} \quad w(1) \le n$$

since no non-empty path from w to $[n,\infty) \times \mathbb{R}$ can have a negative number of vertical sites. Consequently, by iterating (11.33) we obtain for any $\ell \ge 1$

$$(11.34) \qquad P_p\{A(v,n,k)\} \le \sum_{\substack{w_1,y_1 \\ w_1(1)\ge n-M, y_1 \le k}} h(v,w_1,y_1)$$

$$+ \sum_{\substack{w_1,y_1 \\ w_1(1)<n-M, y_1\le k}} h(v,w_1,y_1) \sum_{w_2,y_2} h(v+w_1,w_2,y_2)$$

$$\cdot\ P_p\{A(v+w_1+w_2,n,k-y_1-y_2)\}$$

$$\le \dots \le \sum_{j=1}^{\ell} \sum\nolimits^{(j)} \prod_{i=1}^{j} h(v+w_1+\dots+w_{i-1},w_i,y_i)$$

$$+ \sum_{\substack{w_1(1)+\dots+w_t(1)<n-M \\ \text{for } t\le\ell, \text{ and } y_1+\dots+y_\ell\le k}} \prod_{i=1}^{\ell} h(v+w_1+\dots+w_{i-1},w_i,y_i) ,$$

where $\sum^{(j)}$ is the sum over $w_1,\dots,w_j,y_1,\dots,y_j$ with $w_1(1)+\dots+w_t(1) < n-M$ for $t < j$ but $w_1(1)+\dots+w_j(1) \ge n-M$ and $y_1+\dots+y_j \le k$. Of course all sums in (11.34) are also restricted to $w_i \in [-M-1,M+1] \times [-M-1,M+1]$, $v+w_1+\dots+w_i \in S_1(v+w_1+\dots+w_{i-1},M)$ and $y_i \in \{0,1\}$. Next we take

$$\lambda = \log 16 + 2\log(2M+3)$$

so that, by (11.31)

$$\phi(\lambda) := \max_u \sum_{\substack{u+w\in S_1(u,M) \\ y\in\{0,1\}}} h(u,w,y)e^{-\lambda y}$$

$$\le \max_u \sum_{u+w\in S_1(u,M)} g(u,u+w,M) + e^{-\lambda} 2(2M+3)^2$$

$$\le \frac{3}{4} + 2(2M+3)^2 e^{-\lambda} = \frac{7}{8} .$$

For this choice of λ

$$\sum^{(j)} \prod_{i=1}^{j} h(v+w_1+\ldots+w_{i-1}, w_i, y_i)$$

$$\le e^{\lambda k} \sum_{\substack{y_i \varepsilon \{0,1\} \\ v+w_1+\ldots+w_i \varepsilon S_1(v+w_1+\ldots+w_{i-1}, M)}} \prod_{i=1}^{j} \{e^{-\lambda y_i} h(v+w_1+\ldots+w_{i-1}, w_i, y_i)\}$$

$$\le e^{\lambda k}(\phi(\lambda))^j \le e^{\lambda k}(\tfrac{7}{8})^j ,$$

because $y_1+\ldots+y_j \le k$ in $\sum^{(j)}$. Similarly the last term in (11.34) is at most

$$e^{\lambda k}(\tfrac{7}{8})^{\ell} .$$

Finally, observe that $w_i(1) \le M+1$, so that for $v(1) \le 0$, $v(1)+w_1(1)+\ldots+w_j(1) \ge n-M$ can occur only for $j \ge (n-M)(M+1)^{-1}$. Consequently, by virtue of (11.34)

$$P_p\{A(v,n,k)\} \le \lim_{\ell\to\infty} e^{\lambda k} \sum_{\frac{n-M}{M+1} \le j \le \ell} (\tfrac{7}{8})^j \le 10 e^{\lambda k}(\tfrac{7}{8})^{n/(M+1)} ,$$

whenever $v(1) \le 0$.

If we take

$$k = \frac{1}{2\lambda}(\frac{n}{M+1})\log\frac{8}{7} \sim \{\frac{\alpha_1}{8C_8}\log(\tfrac{8}{7})(p_H-p)^{2/\alpha_1}(-\log(p_H-p))^{-1}\}n,\ p\uparrow p_H ,$$

then we find for $v(1) \le 0$ and suitable C_9

$$P_p\{A(v,n,C_9(p_H-p)^{3/\alpha_1}n)\} \le 10(\tfrac{7}{8})^{\frac{n}{2(M+1)}} \le 10(\tfrac{7}{8})^{\frac{1}{4C_8}(p_H-p)^{2/\alpha_1}n} .$$

This is just (11.27) with $\delta_1 = \frac{3}{\alpha_1}$, $\delta_2 = \frac{2}{\alpha_1}$ and suitable C_6, C_7. □

11.3. Properties of resistances.

Before we start the proof of Theorem 11.2 we describe how the resistance of a network is related to the resistances of the individual edges. As we have done all along we assume that there are no loops (i.e., edges whose endpoints coincide) in the graph. If v is a vertex and e an edge incident to v then we write $w(e,v)$ for the endpoint other than v of e. $R(e)$ is the resistance of the edge e. To find the resistance between the sets of vertices A^0 and A^1 of a finite graph $\mathcal{G}$ one wants to find a potential function $V(\cdot)$ on the vertices of $\mathcal{G}$, which equals 0 on A^0 and one on A^1, and corresponding currents through the edges. The size of the resistance between A^0 and A^1 is then equal to the reciprocal of the total current flowing out of A^0. If v and w are the endpoint of an edge e let $I(v,e)$ denote the current flowing from v to w along e. Ohm's and Kirchhoff's laws (see Feynman et al (1963), Sect. I.25.4,5 and II.22.3, Slepian (1968)) say that the potential and currents have to satisfy

$$I(v,e) = -I(w(e,v),e), \tag{11.35}$$

$$V(w(e,v))-V(v) = R(e)\cdot I(v,e), \tag{11.36}$$

$$\sum_{\substack{e \text{ incident} \\ \text{to } v}} I(v,e) = 0 \quad \text{if} \quad v \notin A^0 \cup A^1 . \tag{11.37}$$

Finally, there is the boundary condition which we imposed

$$V(v) = 0 \quad \text{if} \quad v \in A^0 \quad \text{and} \quad V(v) = 1 \quad \text{if} \quad v \in A^1. \tag{11.38}$$

To discuss the uniqueness and existence of V and I assume first that $R(e) > 0$ for all e. $R(e) = \infty$ is allowed, in which case we have to interpret (11.36) as

$$I(v,e) = 0. \tag{11.39}$$

In this case (11.36) makes no statement about the potential difference between the endpoints of e. However, if $R(e) \neq 0$ we can rewrite (11.36) as

$$I(v,e) = \frac{V(w(e,v))-V(v)}{R(e)} \tag{11.40}$$

and substitution of (11.40) into (11.37) shows that (11.37) is equivalent to

$$V(v) = \frac{\sum_e \frac{V(w(e,v))}{R(e)}}{\sum_e \frac{1}{R(e)}} , \quad v \notin A^0 \cup A^1 , \tag{11.41}$$

where the sums in numerator and denominator of (11.41) run over edges e incident to v. Therefore, $w(e,v)$ runs over all neighbors of v and (11.41) says that $V(v)$ is a weighted average over the values of $V(\cdot)$ at the neighbors of v. $V(w)/R(e)$ and $1/R(e)$ are interpreted as zero if $R(e) = \infty$. Thus, $V(v)$ is really an average of $V(\cdot)$ over those neighbors of v which are connected to v by an edge of finite resistance. (If no such edges exist, then $V(v)$ is not determined by (11.35)-(11.37), but this will turn out to be unimportant.) It is well known from the theory of harmonic functions that the mean value property (11.41) implies a maximum principle; many of the proofs of this fact in the continuous case can be transcribed easily for our situation (see for instance Helms (1969), Theorem 1.12 and also Doyle and Snell (1982), Sect. 2.2). We shall use the following formulation of the maximum principle. Let $\mathcal{C}$ be a collection of vertices disjoint from $A^0 \cup A^1$ and let $\partial_f\mathcal{C}$ denote the set of vertices of $w \notin \mathcal{C}$ which are connected to a vertex in $\mathcal{C}$ by an edge of finite resistance. Then

$$\min_{w\in\partial_f\mathcal{C}} V(w) \le V(v) \le \max_{w\in\partial_f\mathcal{C}} V(w) \tag{11.42}$$

for every vertex v in $\mathcal{C}$ which is connected to $\partial_f\mathcal{C}$ by a conducting path. Here, and in the remainder of this chapter, a <u>conducting path</u> is a path all of whose edges have finite resistance. If we take for $\mathcal{C}$ the set

$$\mathcal{C}_0 = \text{set of all vertices outside } A^0 \cup A^1 ,$$

then (11.42) implies that there is at most one possible value for $V(v)$ at any v connected to $A^0 \cup A^1$ by a conducting path (apply (11.42) to the difference $V'-V''$ of a pair of solutions to (11.35)-(11.39); $V'-V''$ has to vanish on $\partial_f\mathcal{C}_0 \subset A^0 \cup A^1$). The fact that there exists a solution of (11.38) and (11.41) is well known (see Slepian (1968), Ch.7) and is also very easy to prove probabilistically, since $V(v)$ can be interpreted as the probability of hitting A^1 before A^0 when starting at v, for a certain Markov chain (see Doyle and Snell (1982), Sect. 2.7). Once we have $V(\cdot)$, (11.39) and (11.40) give us $I(\cdot)$. The resistance R between A^0 and A^1 is then defined as the reciprocal of the total current flowing out of A^0, i.e.,

$$R = \{\sum_e I(v,e)\}^{-1} = \{\sum_e \frac{V(w(e,v))}{R(e)}\}^{-1} , \tag{11.43}$$

The sums in (11.43) run over all edges e with one endpoint $v \in A^0$ and the other endpoint $w(e,v) \notin A^0$. Note that the right hand side of (11.43) is uniquely determined, despite the fact that $V(w)$ is not unique for a vertex w which is not connected to $A^0 \cup A^1$ by a conducting path. Indeed if $w(e,v)$ is such a vertex, then necessarily $R(e) = \infty$ and the corresponding contribution to (11.43) is zero, no matter how $V(w)$ is chosen.

There are some complications when we allow $R(e) = 0$ for some edges. For such edges e (11.40) is no longer meaningful, and (11.36) merely says

$$V(w(e,v))-V(v) = 0. \tag{11.44}$$

In this case one can proceed as follows. Sum (11.37) over $v \varepsilon \mathcal{C}$ for any set $\mathcal{C}$ disjoint from $A^0 \cup A^1$. If one takes into account that for $v,w \varepsilon \mathcal{C}$ connected by an edge e the sum will contain $I(v,e)+I(w,e) = 0$, one obtains

$$\sum_e I(v,e) = 0, \quad \mathcal{C} \cap (A^0 \cup A^1) = \emptyset, \tag{11.45}$$

where the sum in (11.45) runs over all e with one endpoint $v \varepsilon \mathcal{C}$ and the other endpoint $w(e,v) \notin \mathcal{C}$. In words (11.45) says that the net current flowing out of any set $\mathcal{C}$ disjoint from $A^0 \cup A^1$ must equal zero. Now define two vertices v_1 and v_2 of $\mathcal{G}$ as equivalent iff $v_1 = v_2$ or v_1 and v_2 are connected by a path all of whose edges have zero resistance (i.e., iff there is a short circuit between v_1 and v_2). Let $C_1, C_2, \ldots$ be the equivalence classes of vertices with respect to this equivalence relation. By (11.36), or (11.44), all vertices v in a single class C must have the same potential $V(v)$. We shall write $V(C)$ for this value. Now form a new graph $\mathcal{H}$ by identifying the vertices in an equivalence class. Thus $\mathcal{H}$ has vertex set $\{C_1, C_2, \ldots\}$ and the edges of $\mathcal{H}$ between C_i and C_j are in one to one correspondence with the edges of $\mathcal{G}$ connecting a vertex of C_i with a vertex of C_j. From the above observations we obtain the following analogue for (11.36)-(11.38).

(11.46) $V(C_j)-V(C_i) = R(e)\cdot I(v,e)$, when $v \varepsilon C_i$ is connected to $w \varepsilon C_j$ by e, $i \neq j$.

(11.47) $\sum_e I(v,e) = 0$, wherever $C_i \cap (A^0 \cup A^1) = \emptyset$ and the sum is over all edges e with one endpoint v in C_i and the other endpoint $w(e,v) \notin C_i$.

$$V(C_i) = 0 \quad \text{when} \quad C_i \cap A^0 \neq \emptyset \tag{11.48}$$

$$V(C_i) = 1 \quad \text{when} \quad C_i \cap A^1 \neq \emptyset . \tag{11.49}$$

(11.35) remains unchanged, and (11.46) again is interpreted as (11.39) when $R(e) = \infty$. Of course (11.48) and (11.49) can be simultaneously valid only if there exists no C_i which intersects both A^0 and A^1. However, this case can arise only if A^0 and A^1 are connected by a path all of whose edges have zero resistance. (We shall call such a path a short circuit between A^0 and A^1.) In this case the resistance between A^0 and A^1 is taken to be zero. If there is no C_i which intersects A^0 and A^1, then solving (11.46)-(11.49) just amounts to solving on $\mathcal{H}$ the problem which we solved above for $\mathcal{G}$. We merely have to replace A^0 (A^1) by the collection of C_i which intersect A^0 (A^1). Note that by definition of the C_i any edge from C_i to C_j, $i \neq j$, has a strictly positive resistance. Thus, as before $V(C_i)$ is uniquely determined for any C_i which contains a vertex which is connected to $A^0 \cup A^1$ by a conducting path. For any choice of $V(C_i)$ we then find $I(v,e)$ for $v \in C_i$, e incident to v, but with $w(e,v) \in C_j$ with $j \neq i$. The resistance between A^0 and A^1 in $\mathcal{G}$ can now be defined as the reciprocal of the current flowing out of the union of all C_i which intersect A^0. In analogy to (11.43) this becomes

$$(11.50) \qquad \{\sum I(v,e)\}^{-1} = \{\sum \frac{V(C_j)}{R(e)}\}^{-1} ,$$

where the sum is over all edges e, having one end point in some C_i which intersects A^0 while its other endpoint lies in some C_j with $C_j \cap A^0 = \emptyset$; in the left hand side v is the endpoint of e in C_i, and C_j in the right hand side is the class which contains $w(e,v)$. Just as in (11.43) the sums in (11.50) are uniquely determined.

Even though (11.50) does define the resistance between A^0 and A^1, when $R(e)$ can vanish for some edges e, it would be more intuitive if one could use the middle expression in (11.43) to define the resistance, also in the present case. This is indeed possible, but some more observations are required to see this. The currents between a pair of vertices in the same C_i have not yet been determined, and in fact are not uniquely determined by the equations (11.35)-(11.38). If there are several paths of zero resistance between two vertices there is no reason why the current should be divided into any particular way between these paths. However, there do exist solutions to (11.35)-(11.38). We merely have to take

(11.51) $$V(v) = V(C_i) \quad \text{when} \quad v \in C_i ,$$

where $V(C_i)$ satisfies (11.46)-(11.49). Then (11.38) is automatically true. Next define

(11.52) $$I(v,e) = \frac{V(w(e,v))-V(v)}{R(e)} = \frac{V(C_j)-V(C_i)}{R(e)} \quad \text{when } e \text{ is an edge with endpoints } v \in C_i \text{ and } w(e,v) \in C_j \text{ with } i \neq j.$$

(11.52) makes sense since $R(e) \neq 0$ for an edge e with endpoints in different C_i and C_j. (11.51) and (11.52) guarantee that (11.36) holds, no matter how we choose $I(v,e)$ for an edge e with both endpoints in the same C_i and (11.35) is already satisfied for v and $w(e,v)$ in different C_i and C_j. We now merely have to choose $I(v,e)$ for edges e with both endpoints in one C_i, in such a way that (11.35) and (11.37) hold. (11.37) can be written as

(11.53) $$\sum_{\substack{e \text{ such that} \\ w(e,v) \in C_i}} I(v,e) = - \sum_{\substack{e \text{ such that} \\ w(e,v) \notin C_i}} I(v,e), \quad \text{for } v \in C_i \setminus A^0 \cup A^1 ,$$

for all C_i which contain at least two vertices. The right hand side of (11.53) has already been determined in (11.52). To satisfy (11.35) we choose for each edge e with endpoints v, w in C_i one of the endpoints as the first one, v say. Then we take $I(v,e)$ as an independent variable and set $I(w,e) = -I(v,e)$. Then for each C_i which does not intersect $A^0 \cup A^1$ the expressions obtained in the left hand side of (11.53) as v ranges over C_i, contain each independent variable exactly twice, once with coefficient +1 and once with coefficient -1. Thus, the sum of the left hand side of (11.53) over v in C_i vanishes. The same is true for the sum of the right hand sides, by virtue of (11.47) if $C_i \cap (A^0 \cup A^1) = \emptyset$. By induction on the number of variables one easily sees that (11.53) has at least one solution which satisfies (11.35), and (11.37). If C_i intersects $A^0 \cup A^1$, then rewrite (11.37) or (11.53) as

(11.54) $$\sum_{\substack{e \text{ such that} \\ w(e,v) \in C_i \setminus A^0 \cup A^1}} I(v,e) = - \sum_{\substack{e \text{ such that} \\ w(e,v) \notin C_i}} I(v,e)$$

$$- \sum_{\substack{e \text{ such that} \\ w(e,v) \varepsilon C_i \cap (A^0 \cup A^1)}} I(v,e), \quad v \varepsilon C_i \setminus (A^0 \cup A^1).$$

The above argument used for (11.53) now shows that (11.54) has a solution satisfying (11.35) if and only if the sum of the right hand sides of (11.54) over $v \varepsilon C_i \setminus (A^0 \cup A^1)$ vanishes, i.e., iff we choose $I(v,e)$ for $v \varepsilon C_i \setminus A^0 \cup A^1$, and $w(e,v) \varepsilon C_i \cap (A^0 \cup A^1)$ such that

$$(11.55) \qquad - \sum_{v \varepsilon C_i \setminus (A^0 \cup A^1)} \sum_{\substack{e \text{ such that} \\ w(e,v) \varepsilon C_i \cap (A^0 \cup A^1)}} I(v,e)$$

$$= \sum_{w \varepsilon C_i \cap (A^0 \cup A^1)} \sum_{\substack{e \text{ such that} \\ v(e,w) \varepsilon C_i \setminus (A^0 \cup A^1)}} I(w,e)$$

$$= \sum_{v \varepsilon C_i \setminus (A^0 \cup A^1)} \sum_{\substack{e \text{ such that} \\ w(e,v) \notin C_i}} I(v,e).$$

Thus we can always solve (11.35)-(11.38), and any solution has to be chosen such that (11.55) holds.

Now that we have shown that there is a solution to (11.35)-(11.38) we can show that the left hand side of (11.50) equals the middle expression in (11.43) by the following general argument. Let $\mathcal{D}$ be any set of vertices which contains A^0 and is disjoint from A^1. We claim that for any such $\mathcal{D}$

$$(11.56) \qquad R = \{\textstyle\sum_{\mathcal{D}} I(v,e)\}^{-1} = \{\textstyle\sum_{A^0} I(v,e)\}^{-1} ,$$

where $\sum_{\mathcal{D}}$ runs over all edges e with one endpoint $v \varepsilon \mathcal{D}$ and the other endpoint $w(e,v)$ outside $\mathcal{D}$. In accordance with this notation, the last member of (11.56) is just the middle member of (11.43) $\sum_{\mathcal{D}} I(v,e)$ represents the current flowing out of $\mathcal{D}$. To prove (11.56) we apply (11.45) with $\mathcal{C} = \mathcal{D} \setminus A^0$. By our choice of $\mathcal{D}$, $\mathcal{C}$ is disjoint from $A^0 \cup A^1$. (11.45) can now be rewritten as

$$(11.57) \qquad \textstyle\sum_1 I(v,e) = - \sum_1 I(w(e,v),e) = \sum_2 I(v,e)$$

where $\sum_1$ runs over all edges e with one endpoint v in A^0 and the other endpoint $w(e,v)$ in $\mathcal{D}\setminus A^0$, while $\sum_2$ runs over all edges e with one endpoint v in $\mathcal{D}\setminus A^0$ and the other endpoint $w(e,v)$ outside $\mathcal{D}$. Now add to both sides of (11.57) the sum $\sum_3 I(v,e)$ over all edges e with one endpoint v in A_0 and the other endpoint $w(e,v)$ outside $\mathcal{D}$. Then

$$\sum_1 I(v,e) + \sum_3 I(v,e)$$

is just the sum in the middle member of (11.43). On the other hand

$$\sum_2 I(v,e) + \sum_3 I(v,e)$$

is just $\sum_{\mathcal{D}} I(v,e)$, so that (11.56) follows.

We return to (11.50). Take $\mathcal{D} = \cup C_i$ where the union is over all C_i which intersect A^0. We ruled out the case in which some C_i intersects A^0 and A^1; we took the resistance between A^0 and A^1 zero in this case. With this case ruled out we see that $\mathcal{D} = \cup C_i$ is indeed disjoint from A^1 so that (11.56) applies. But for this $\mathcal{D}$, $\sum_{\mathcal{D}}$ is just the sum in the left hand side of (11.50). Thus the middle member of (11.43) can be used to define R, even if $R(e)$ can vanish for some e (as long as there is no short circuit between A^0 and A^1).

It is worth pointing out that for a finite graph $\mathcal{G}$ the above definition of R implies

(11.58) $R = 0$ if and only if there is a short circuit between A^0 and A^1

and

(11.59) $R = \infty$ if and only if there is no conducting path from A^0 to A^1.

(11.58) is immediate from (11.56), since we have taken all currents finite, as long as there is no path from A^0 to A^1 with all its edges of zero resistance. For (11.59), assume first that $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ is a conducting path from A^0 to A^1, i.e., with $v_0 \in A^0$, $v_\nu \in A^0$. By going over to a subpath we may assume $v_i \notin A^0 \cup A^1$ for $1 \le i \le \nu-1$. Of course r is also self-avoiding. Then $\sum\{V(v_{i+1})-V(v_i)\} = V(v_\nu)-V(v_0) = 1$ by (11.38)). Thus for some j $V(v_{j+1})-V(v_j) > 0$, and since $R(e_{j+1}) < \infty$ also $I(v_j,e_{j+1}) > 0$. If we take $\mathcal{D} = A_0 \cup \{v_0,\dots,v_j\}$ then the middle sum in (11.56) contains the term $I(v_j,e_{j+1}) > 0$, so that $R < \infty$. Conversely, if $R < \infty$, take $\mathcal{D} = A^0 \cup \{v\colon v \text{ is connected to } A^0 \text{ by a conducting path}\}$.

Then $\mathcal{D}$ must contain a vertex on A^1. Indeed all edges e with one endpoint v in $\mathcal{D}$ and the other endpoint w(e,v) outside $\mathcal{D}$ must have infinite resistance (otherwise we should add w(e,v) to $\mathcal{D}$). But then $\sum_{\mathcal{D}} I(v,e) = 0$, which contradicts $R < \infty$, unless $\mathcal{D} \cap A^1 \neq \emptyset$. But $\mathcal{D}$ intersects A^1 if and only if there is a conducting path from A^0 to A^1. This proves (11.59).

With these preparations it is not hard to prove the following four lemmas. The first two and the fourth lemma are intuitively obvious from their electrical interpretation and reading of their proofs should be postponed. In all four lemmas we take the assignment of resistances to the edges as fixed, i.e., non-random.

Lemma 11.1. *Let $\mathcal{G}$ be a finite planar graph and R the resistance between two disjoint subsets A^0 and A^1 of $\mathcal{G}$. Assume that e_0 is an edge of $\mathcal{G}$ such that $\mathring{e}_0$ is surrounded by a circuit $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ (with $v_i \neq v_j$, $i \neq j$, except for $v_0 = v_\nu$) with*

$$R(e_i) = 0, \; 1 \leq i \leq \nu ,$$

and such that $A^0 \cup A^1$ contains no vertex in the interior of r. Then R is unchanged if $R(e_0)$ is replaced by zero.

Proof: As before we may exclude the case in which A^0 and A^1 are connected by a path of zero resistance. In that case $R = 0$ and this is even more true when $R(e)$ is set equal to zero. Thus in this case there is nothing to prove. In the other case let $V(r)$ be the potential at v when all vertices of A^0 (A^1) are given the potential zero (one). R can then be calculated as the resistance between A^0 and A^1 on the graph $\mathcal{K}$ obtained by identifying vertices in a single equivalence class C_i all of whose vertices are connected by paths of zero resistance; see the argument preceding (11.50). By assumption all vertices on r will belong to one such equivalence class, say C_{i_0}. Moreover, the vertices inside the circuit r either belong to C_{i_0} or to some C_j which can be connected to $A^0 \cup A^1$ only via C_{i_0} (by the planarity of $\mathcal{G}$). Let $C_{j_1}, \dots, C_{j_\lambda}$ be the classes other than C_{i_0} which contain vertices inside r. Then all vertices of $\mathcal{G}$ in $\bigcup_{\alpha=1}^{\lambda} C_{j_\alpha}$ lie in int(r). The boundary on $\mathcal{K}$ of the set of vertices $\{C_{j_1}, \dots, C_{j_\lambda}\}$ of $\mathcal{K}$ is the one point C_{i_0} (see Def. 2.8). By the maximum principle (11.42) applied to $\mathcal{K}$ it follows that

(11.60) $$V(C_{j_\alpha}) = V(C_{i_0})$$

for all C_{j_α} which are connected to C_{i_0} by a conducting path, $1 \le \alpha \le \lambda$. We shall now show that this implies that R is unchanged if all vertices in the interior of r are removed from $\mathcal{G}$. Then $C_{i_1},\ldots,C_{i_\lambda}$ are removed from $\mathcal{K}$, and C_{i_0} becomes the class of vertices on r or outside r, but connected to r by a path of zero resistance. To see that this removal does not effect R note that (11.46)-(11.49) remain satisfied with the values of $V(C_i)$ and $I(v,e)$ unchanged as long as i and j are restricted to the complement of $\{j_1,\ldots,j_\lambda\}$, and of course e such that its endpoints v and w do not belong to the interior of r. This is so because an edge from some $v \in C_i$ with $i \notin \{j_1,\ldots,j_\lambda\}$ and $v \notin \text{int}(r)$ to some $w \in \text{int}(r)$ can exist only if v is a vertex on r and hence $v \in C_{i_0}$. In this case $I(v,e)$ is zero anyway, either because $R(e) = \infty$, or by virtue of (11.60). Thus the term corresponding to e be dropped from (11.47) without changing the left hand side of (11.47). But, then the right hand side of (11.50) does not change either when the vertices inside r are removed, again because $C_{j_1},\ldots,C_{j_\lambda}$ do not contribute to the sum in the right hand side of (11.50). Indeed if C_{j_α} contains a vertex w connected to some $v \in A_0$ by an edge e and $C_{j_\alpha} \cap A^0 = \emptyset$, then v must belong to C_{i_0}, and $V(C_{i_0}) = 0$ by (11.48). The term $V(C_{j_\alpha})/R(e)$ again vanishes, either because $R(e) = \infty$ or because of (11.60). Thus R is indeed unchanged if all vertices in the interior of r are removed, and this has been proven without any reference to the value of $R(e_0)$. Since e_0 is no longer part of the network after removal of the vertices in $\text{int}(r)$, the value of R is independent of $R(e_0)$. □

<u>Lemma 11.2.</u> <u>Let</u> $\mathcal{G}$, A^0, A^1 <u>and</u> R <u>be as in Lemma 11.1, and let</u>

$$\mathcal{C}_1 = A^0 \cup A^1 \cup \{v \in \mathcal{G} : v \text{ is connected to } A^0 \text{ by a conducting path}\}.$$

<u>Then</u> R <u>is unchanged if we replace</u> $R(e)$ <u>by infinity for each edge</u> e <u>which does not have both endpoints in</u> $\mathcal{C}_1$.

Proof: Since the current is zero in any edge with infinite resistance, the right hand sides of (11.43) and (11.50) involve only the values of $V(w)$ with $w \in \mathcal{C}_1$ - terms which do not involve such w give zero contributions. It therefore suffices to show that $V(\cdot)$ is uniquely determined on $\mathcal{C}_1$, and that its values on $\mathcal{C}_1$ are unchanged if we take $R(e) = \infty$ for each edge e which does not have both endpoints in $\mathcal{C}_1$. Let us first assume $R(f) > 0$ for all edges f. Then the restriction of $V(\cdot)$ to $\mathcal{C}_1$ satisfies (11.41) and (11.38) on $\mathcal{C}_1$. Moreover, for $v \in \mathcal{C}_1$, an edge e incident to v only gives a non-zero contribution to the sums in the right hand side of (11.41) if its second endpoint $w(e,v)$ also lies in $\mathcal{C}_1$. Consequently we can view the restriction of $V(\cdot)$ to $\mathcal{C}_1$ as the solution for the potential on the graph $\mathcal{G}_{|\mathcal{C}_1}$ say, whose vertex set is $\mathcal{C}_1$ and whose edge set is the set of edges of $\mathcal{G}$ between two points of $\mathcal{C}_1$. Under (11.38) this problem on $\mathcal{G}_{|\mathcal{C}_1}$ has a unique solution for the same reasons as in the original problem on $\mathcal{G}$. (Note that the maximum principle (11.42) only involves edges of $\mathcal{G}_{|\mathcal{C}_1}$ when $C \subset \mathcal{C}_1$.) Thus $V_{|\mathcal{C}_1}$ does not change if we change the resistance of edges which do not have both endpoints in $\mathcal{C}_1$, provided we do not change $\mathcal{C}_1$. In particular $\mathcal{C}_1$ does not change if we set $R(e) = \infty$ for some of these edges. This proves the lemma if all edges have a strictly positive resistance.

To prove the lemma when some $R(e)$ may vanish we merely have to apply the preceding argument to the graph $\mathcal{H}$ whose vertices are the equivalence classes C_j introduced after (11.45). (Note that if C_j has any point in $\mathcal{C}_1$, then $C_j \subset \mathcal{C}_1$.) $\square$

For the next lemma we remind the reader that as a planar graph, $\mathbb{Z}^2$ has a dual graph, $(\mathbb{Z}^2)_d$ (see Sect. 2.6, especially Ex. 2.6(i)). $(\mathbb{Z}^2)_d$ can be thought of as the graph with vertices at $(i+\frac{1}{2}, j+\frac{1}{2})$, $i,j \in \mathbb{Z}$, and $(i_1+\frac{1}{2}, j_1+\frac{1}{2})$, $(i_2+\frac{1}{2}, j_2+\frac{1}{2})$ connected by an edge if and only if $|i_1-i_2| + |j_1-j_2| = 1$. Each edge e^* of $(\mathbb{Z}^2)_d$ intersects exactly one edge e of $\mathbb{Z}^2$ and vice versa. If e and e^* are associated in this manner we shall assign to e^* the resistance $S(e^*) = 1/R(e)$. In this way, each assignment of resistances on $\mathbb{Z}^2$ induces a unique assignment of resistances on $(\mathbb{Z}^2)_d$. Finally, let $R([a_1,a_2]\times[b_1,b_2])$ $(S([a_1,a_2]\times[b_1,b_2]))$ be the resistance between the left and right edge (top and bottom edge) of $[a_1,a_2]\times[b_1,b_2]$ of the network consisting of the edges of $\mathbb{Z}^2$ $((\mathbb{Z}^2)_d)$ in $[a_1,a_2]\times[b_1,b_2]$.

Lemma 11.3 which is taken from Straley (1977) gives a duality relation between resistances on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)_d$.

Lemma 11.3. *If* $S(e^*) = 1/R(e)$ *for all pairs of edges* e *of* $\mathbb{Z}^2$ *and* e^* *of* $(\mathbb{Z}^2)_d$ *which intersect, then for integral* $a_1 < a_2$, $b_1 < b_2$,

$$R([a_1,a_2]\times[b_1,b_2]) = \{S([a_1+\tfrac{1}{2},a_2-\tfrac{1}{2}]\times[b_1-\tfrac{1}{2},b_2+\tfrac{1}{2}])\}^{-1} . \tag{11.61}$$

Proof: For the time being assume

$$0 < R([a_1,a_2]\times[b_1,b_2]) < \infty . \tag{11.62}$$

Let $V(v)$ denote the potential at v and $I(v,e)$ the current from v to $w(e,v)$ along e in the network consisting of the restriction of $\mathbb{Z}^2$ to $[a_1,a_2]\times[b_1,b_2]$ when all vertices on $A^0 := \{a_1\}\times[b_1,b_2]$ are given potential zero, and all vertices on $A^1 := \{a_2\}\times[b_1,b_2]$ are given potential one. As explained, $V(\cdot)$ and $I(\cdot,\cdot)$ have to satisfy (11.35)-(11.38) on $[a_1,a_2]\times[b_1,b_2]$. Even though this may not uniquely determine V and I, $R([a_1,a_2]\times[b_1,b_2])$ is uniquely given by (11.43), with $\mathcal{G}$ = restriction of $\mathbb{Z}^2$ to $[a_1,a_2]\times[b_1,b_2]$. We first extend $V(\cdot)$ also to the points $[a_1+1,a_2-1]\times\{b_1-1\}$ and $[a_1+1,a_2-1]\times\{b_2+1\}$, just below and just above $[a_1,a_2]\times[b_1,b_2]$. We do this by setting

$$V((i,b_1-1)) = V((i,b_1)),\ V((i,b_2+1)) = V((i,b_2)), \tag{11.63}$$
$$a_1+1 \le i \le a_2+1 .$$

To maintain (11.35)-(11.37) we also set

(11.64) $I(v,e) = I(w,e) = 0$ when $v = (i,b_1-1)$, $w = (i,b_1)$ or $v = (i,b_2+1)$, $w = (i,b_2)$, and e the edge between v and w.

Now let e and e^* be a pair of edges of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)_d$, respectively, which intersect in their common midpoint m. Let the endpoints of e be v and w. When e is rotated counterclockwise over an angle $\frac{\pi}{2}$, then e goes over into e^*. Let v (w) go over into v^* (w^*) under this rotation (see Fig. 11.2 for some illustrations). We then set for $e^* \subset [a_1+\frac{1}{2},a_2-\frac{1}{2}]\times[b_1-\frac{1}{2},b_2+\frac{1}{2}]$

$$J(v^*,e^*) = -J(w^*,e^*) = V(w)-V(v) \tag{11.65}$$

$$W(w^*)-W(v^*) = I(v,e), \tag{11.66}$$

Figure 11.2

(11.67) $\qquad W(v^*) = 0 \quad \text{for} \quad v^* = (i+\frac{1}{2}, b_1 - \frac{1}{2}),\ a_1 \le i \le a_2 - 1.$

We claim that (11.65)-(11.67) define a potential W and current J on the network $(\mathbb{Z}^2)_d$ restricted to $[a_1+\frac{1}{2}, a_2-\frac{1}{2}] \times [b_1-\frac{1}{2}, b_2+\frac{1}{2}]$. Moreover

(11.68) $\qquad W(\cdot) = 0 \quad \text{on} \quad [a_1+\frac{1}{2}, a_2-\frac{1}{2}] \times \{b_1-\frac{1}{2}\}$

and $\quad W(\cdot) = \frac{1}{R} \quad \text{on} \quad [a_1+\frac{1}{2}, a_2-\frac{1}{2}] \times \{b_2+\frac{1}{2}\}$,

where $R = R([a_1,a_2] \times [b_1,b_2])$. To substantiate this claim we must first show that (11.66) and (11.67) are consistent and define the function $W(\cdot)$ unambiguously. First, we obtain from (11.64) that if e^* is the edge from $v^* = (i_1-\frac{1}{2}, b_1-\frac{1}{2})$ to $w^* = (i_1+\frac{1}{2}, b_1-\frac{1}{2})$ which intersects the edge e from $v = (i_1, b_1-1)$ to $w = (i_1, b_1)$ then $I(v,e) = 0$. Hence (11.66) tells us to take $W(w^*) = W(v^*)$ in this case. This is in agreement with the constancy of $W(\cdot)$ on $[a_1+\frac{1}{2}, a_2-\frac{1}{2}] \times \{b_1-\frac{1}{2}\}$ as required by (11.67). Next we must verify that if $r^* = (v_0^*, e_1^*, \dots, e_\nu^*, v_\nu^*)$ (with $v_\nu^* = v_0^*$) is a simple closed path on $(\mathbb{Z}^2)_d$ restricted to $[a_1+\frac{1}{2}, a_2-\frac{1}{2}] \times [b_1-\frac{1}{2}, b_2+\frac{1}{2}]$, then

$$\sum_{i=0}^{\nu-1} \{W(v_{i+1}^*) - W(v_i^*)\}$$

as defined by (11.66) indeed has the value zero. In other words, if v_i^* (v_{i+1}^*) is the image of v_i (w_i) after rotating the edge e_{i+1} from v_i to w_i counterclockwise over $\frac{\pi}{2}$ around the common midpoint of e_{i+1}^* and e_{i+1}, then we must show

(11.69) $$\sum_{i=0}^{\nu-1} I(v_i, e_{i+1}) = 0.$$

Once we prove this we can define $W(v^*)$ as

$$W(v^*) = \sum_{i=0}^{\lambda-1} \{W(v^*_{i+1})-W(v^*_i)\} = \sum_{i=0}^{\lambda-1} I(v_i,w_i) \tag{11.70}$$

for any path $(v^*_0,e^*_1,\dots,e^*_\lambda,v^*_\lambda)$ in $[a_1+\frac{1}{2},a_2-\frac{1}{2}]\times[b_1-\frac{1}{2},b_2+\frac{1}{2}]$ with v^*_0 on $[a_1+\frac{1}{2},a_2-\frac{1}{2}]\times\{b_1-\frac{1}{2}\}$ and $v^*_\lambda = v^*$; all the sums in (11.70) will have the same value. To prove (11.69) whenever $r = (v^*_0,e^*_1,\dots,e^*_\nu,v^*_\nu)$ is a simple closed curve is easy. Since the interior of r^* is the union of a finite number of unit squares of the form $(c-\frac{1}{2},c+\frac{1}{2})\times(d-\frac{1}{2},d+\frac{1}{2})$ it follows from standard topological arguments (see Newman (1951), Ch.V.1-V.5, especially Theorem V.21) that it suffices to verify (11.69) if r^* describes the perimeter of such a unit square. Thus, it suffices to take $\nu = 4$, $v^*_0 = v^*_4 = (c-\frac{1}{2},d-\frac{1}{2})$, $v^*_1 = (c+\frac{1}{2},d-\frac{1}{2})$, $v^*_2 = (c+\frac{1}{2},d+\frac{1}{2})$, $v^*_3 = (c-\frac{1}{2},d+\frac{1}{2})$ (see Fig. 11.3) and e^*_{i+1} = the edge from v^*_i to v^*_{i+1}. However, in this case e^*_{i+1} is obtained by rotating the edge e_{i+1} from v to w_i counterclockwise over $\frac{\pi}{2}$, where

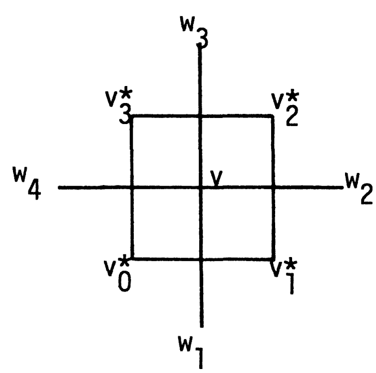

Figure 11.3

$v = (c,d)$, and $w_1,\dots,w_4$ runs over the four neighbors of v. Thus (11.69) reduces to

$$\sum_{\substack{e \text{ incident} \\ \text{to } v \text{ on } \mathbb{Z}^2}} I(v,e) = 0 ,$$

which is just Kirchhoff's law (11.37). Thus (11.69) holds, and $W(\cdot)$ is well defined. It satisfies the first relation in (11.68) by virtue of (11.67). The second rotation in (11.68) is verified as follows.

For $v^* = (i_1+\frac{1}{2}, b_2+\frac{1}{2})$ and e_j the edge from (i_1,b_1+j) to (i_1+1,b_1+j) we obtain from (11.70)

$$\text{(11.71)} \qquad W(v^*) = \sum_{j=0}^{b_2-b_1} I((i_1,b_1+j),e_j) ,$$

which is the total current flowing from left to right through the segment $C = \{i_1\}\times[b_1,b_2]$ in $\mathbb{Z}^2 \cap [a_1,a_2]\times[b_1,b_2]$. This is precisely $1/R$ when $i_1 = a_1$, by definition of R (see (11.43)). However, (11.45) shows that if e_j' denotes the edge from (i_1,b_1+j) to (i_1-1,b_1+j), then

$$\sum_{j=0}^{b_2-b_1} I((i_1,b_1+j),e_j) + \sum_{j=0}^{b_2-b_1} I((i_1,b+j),e_j') = 0, \ a_1 < i_1 < a_2.$$

Since by (11.35) $I((i_1,b_1+j),e_j') = -I((i_1-1,b+j),e_j')$ this says that (11.71) has the same value for all $a_1 \le i_1 < a_2$. (Intuitively, this merely says that the total current flowing into C from the left equals the total current flowing out to the right from C.) This proves (11.68).

It is also obvious that

$$W(w^*(e^*,v))-W(v^*) = I(v,e) = \frac{V(w(e,v))-V(v)}{R(e)} = S(e^*)J(v^*,e^*),$$

when e^* intersects e by (11.66), (11.36) and (11.65), provided $R(e) \neq 0$. Thus in this case W and J satisfy the analogue of (11.36). If $R(e) = 0$ then $S(e^*) = \infty$ and in this case the analogue of (11.36) requires $J(v^*,e^*) = 0$ (see (11.39)). This is also satisfied, since $R(e) = 0$ implies $V(w(e,v))-V(v) = 0$ by (11.36) and then $J(v^*,e^*) = 0$ by (11.65). Thus W and J satisfy Ohm's law. Finally, we must verify Kirchhoff's law for J, i.e.,

$$\text{(11.72)} \qquad \sum_{\substack{e^* \text{ incident} \\ \text{to } v^* \text{ on } (\mathbb{Z}^2)_d}} J(v^*,e^*) = 0, \text{ for}$$

$$v^* \in [a_1+\tfrac{1}{2}, a_2-\tfrac{1}{2}]\times(b_1+\tfrac{1}{2}, b_2-\tfrac{1}{2}).$$

If $v^* = (c+\frac{1}{2}, d+\frac{1}{2})$ with $a_1 < c < a_2$, then by (11.65), (11.72) simply reduces to the relation

$$\sum_{i=0}^{3}\{V(v_{i+1})-V(v_i)\} = 0,$$

where $v_0 = v_4 = (c+1,d)$, $v_1 = (c,d)$, $v_2 = (c,d+1)$, $v_3 = (c+1,d+1)$.

This relation trivially holds since $v_0 = v_4$. If $c = a_1$, then there is no edge in our network between $(c+\frac{1}{2},d+\frac{1}{2})$ and $(c-\frac{1}{2},d+\frac{1}{2})$. Thus the term $\{V(v_2)-V(v_1)\} = V((c,d+1))-V((c,d)) = V((a_2,d+1))-V((a_1,d))$ has to be dropped from the last sum. However, this term is zero anyway, by virtue of (11.38). Thus (11.72) remains valid even when $c = a_1$, and a similar argument applies when $c = a_2$. Thus J and W satisfy the analogues of (11.35)-(11.38) as desired. ((11.35) is trivial from (11.65).) Thus $RJ(\cdot)$ can be taken as the current in $(\mathbb{Z}^2)_d \cap [a_1+\frac{1}{2},a_2-\frac{1}{2}] \times [b_1-\frac{1}{2},b_2+\frac{1}{2}]$ when the potential on $[a_1+\frac{1}{2},a_2-\frac{1}{2}] \times \{b_1-\frac{1}{2}\}$ is set equal to zero and on $[a_1+\frac{1}{2},a_2-\frac{1}{2}] \times \{b_2+\frac{1}{2}\}$ equal to one, and when e^* has resistance $S(e^*) = 1/R(e)$. Consequently, by the definition (11.43) and (11.65)

$$S([a_1+\tfrac{1}{2},a_2-\tfrac{1}{2}] \times [b_1-\tfrac{1}{2},b_2+\tfrac{1}{2}])$$

$$= \{\sum_{a_1 \le i < a_2} RJ((i+\tfrac{1}{2},b_1-\tfrac{1}{2}),(i+\tfrac{1}{2},b_1+\tfrac{1}{2}))\}^{-1}$$

$$= \frac{1}{R}\{\sum_{a_1 \le i < a_2} (V((i+1,b_1)-V(i,b_1))\}^{-1}$$

$$= \frac{1}{R}\{V(a_2,b_1)-V(a_1,b_1)\}^{-1} .$$

Since V was taken zero (one) on $\{a_1\} \times [b_1,b_2](a_2 \times [b_1,b_2])$ this proves (11.61) whenever (11.62) holds.

By (11.58) the case $R([a_1,a_2] \times [b_1,b_2]) = 0$ occurs if and only if there exists a path $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ on $\mathbb{Z}^2 \cap [a_1,a_2] \times [b_1,b_2]$ from A^0 to A^1 such that $R(e_i) = 0$ for all $1 \le i \le \nu$. Let $\mathfrak{D}$ be the part of $[a_1+\frac{1}{2},a_2-\frac{1}{2}] \times [b_1-\frac{1}{2},b_2+\frac{1}{2}]$ below r. Each edge of $(\mathbb{Z}^2)_d \cap [a_1+\frac{1}{2},a_2-\frac{1}{2}] \times [b_1-\frac{1}{2},b_2+\frac{1}{2}]$ leaving $\mathfrak{D}$ has to be an edge e^* which intersects some e_i, $1 \le i \le \nu$, and therefore has $S(e^*) = \infty$. Thus, no current can leave $\mathfrak{D}$ in the network $(\mathbb{Z}^2)_d \cap [a_1+\frac{1}{2},a_2-\frac{1}{2}] \times [b_1-\frac{1}{2},b_2+\frac{1}{2}]$. Thus by (11.56) applied to this network $S([a_1+\frac{1}{2},a_1-\frac{1}{2}] \times [b_1-\frac{1}{2},b_2+\frac{1}{2}]) = \infty$. Thus (11.61) also holds if $R = 0$.

Finally, if $R([a_1,a_2] \times [b_1,b_2]) = \infty$, then by (11.59) there is no conducting path from A^0 to A^1 in $[a_1,a_2] \times [b_1,b_2]$. By Prop. 2.2, or somewhat more directly by Whitney's theorem (see Smythe and Wierman (1978), proof of Theorem 2.2) it follows that there is then a path r^* on $(\mathbb{Z}^2)_d \cap [a_1 + \frac{1}{2}, a_2 - \frac{1}{2}] \times [b_1 - \frac{1}{2}, b_2 + \frac{1}{2}]$ all of whose edges have zero resistance and which connects the bottom edge $[a_1 + \frac{1}{2}, a_2 - \frac{1}{2}] \times \{b_1 - \frac{1}{2}\}$ with the top edge $[a_1 + \frac{1}{2}, a_2 - \frac{1}{2}] \times \{b_2 + \frac{1}{2}\}$. (Prop. 2.2 is somewhat clumsier than Whitney's theorem here, because it requires transference of the problem to the covering graph.) The existence of r^* shows that $S([a_1 + \frac{1}{2}, a_2 - \frac{1}{2}] \times [b_1 - \frac{1}{2}, b_2 + \frac{1}{2}]) = 0$. This confirms (11.61) in the last case. □

The reader probably does not need to be reminded that when two vertices v' and v'' are connected only by a path $(v' = v_0, e_1, \dots, e_\nu, v_\nu = v'')$ then the resistance between v' and v'' is $\sum_1^\nu R(e_i)$. (The resistances $R(e_1), \dots, R(e_\nu)$ are in series in this case; see Feynman et al (1963), Sect. I.25.5 and II.22.3.) Also, if v' and v'' are connected exactly by k paths $r_1, \dots, r_k$ which are pairwise disjoint (except for the common endpoints v' and v'') then these paths form parallel resistances. If the edges in r_j are $\{e_{ji}\}$, then the resistance of r_j is $\sum_i R(e_{ji})$ and the resistance between v' and v'' is

$$\Big(\sum_j \ \{\sum_i R(e_{ji})\}^{-1}\Big)^{-1}$$

(see Feynman et al (1963), Sect. I.25.5 and II.22.3).

Finally we make repeated use of the following monotonicity property.

Lemma 11.4. *If $R_1(e)$ and $R_2(e)$ are two assignments of resistances to the edges of $\mathcal{G}$ and R_1, R_2 the corresponding values of the resistance between A^0 and A^1, then*

(11.73) $$R_1(e) \le R_2(e) \text{ for all } e \text{ implies } R_1 \le R_2.$$

Proof: Despite its intuitive content, we have no intuitive proof of (11.73). A quick proof for (11.73) when

(11.74) $$0 < R_1(e) \le R_2(e) \text{ for all } e$$

can be found in Griffeath and Liggett (1983), Theorem 2.1. It is based on the fact that under (11.74) the expression

$$(R_i)^{-1} = \sum_e \frac{V_i(w(e,v))}{R(e)}$$

given in (11.43) for the reciprocal of R_i can also be written as

$$\frac{1}{2}\min_h \sum_{e\varepsilon\mathcal{G}} \frac{\{h(v)-h(w)\}^2}{R_i(e)}, \quad i = 1,2. \tag{11.75}$$

Here $V_i(\cdot)$ is the potential function corresponding to the resistances $R_i(e)$, v and w denote the endpoints of e, while the min in (11.75) is over all functions h from the vertex set of $\mathcal{G}$ into $[0,1]$ which satisfy

$$h(v) = 0 \text{ if } v \varepsilon A^0, \; h(v) = 1 \text{ if } w \varepsilon A^1.$$

(11.75) is usually called Dirichlet's principle. Its proof works as long as all edges have a strictly positive resistance; edges with infinite resistance do not cause difficulties. Clearly, (11.73) follows immediately from (11.75) whenever (11.74) holds[1]. When $R_1(e)$ and/or $R_2(e)$ can be zero we have to use a limiting procedure. Let

$$R^\varepsilon(e) = R(e)+\varepsilon, \; R_i^\varepsilon(e) = R_i(e)+\varepsilon$$

and denote by R^ε, R_i^ε the corresponding resistance between A^0 and A^1. Then by (11.73) $R_1^\varepsilon \le R_2^\varepsilon$ for all $\varepsilon > 0$. It therefore suffices to show that

$$R^\varepsilon \downarrow R^0 = R \text{ as } \varepsilon \downarrow 0. \tag{11.76}$$

(11.76) is very easy if there exists a short circuit between A^0 and A^1. For in this case we defined R as zero, while

> $R^\varepsilon \le \varepsilon$ (number of edges in any path from A^0 to A^1 all of whose edges have zero resistance)

(apply (11.73) with $R_1(e) = R^\varepsilon(e)$ and $R_2(e) = R^\varepsilon(e)$ for e belonging to some short circuit between A^0 and A^1 and $R_2(e) = \infty$ otherwise).

For the remainder of this lemma assume that there is no short circuit between A^0 and A^1. Observe that by the maximum principle (11.42) $0 \le V^\varepsilon(v) \le 1$ for all v in the set C_1 of Lemma 11.2. (Of course V^ε and I^ε denote the potential and current when $R^\varepsilon(e)$ is

---

[1] An alternative approach to (11.73) is via Thomson's principle, which is a dual to Dirichlet's principle (see Doyle and Snell (1982), Sect. 2.9). This works well as long as all edges have finite resistance. In this approach one has to prove an analogue of (11.76) as resistances increase to ∞.

the resistance of e, and the boundary condition (11.38) is imposed.) Note that $\mathcal{C}_1$ is the same for all ε and that we showed in the proof of Lemma 11.2 that R^{ε} is determined by $V_{|\mathcal{C}_1}$. Now let ε_n be any sequence decreasing to 0. Since $V^{\varepsilon}_{|\mathcal{C}_1}$ is uniformly bounded we can pick a subsequence (which for convenience we still denote by $\{\varepsilon_n\}$) such that $V^{\varepsilon}_{|\mathcal{C}_1}$ converges to some function $\tilde{V}$ on $\mathcal{C}_1$ as ε runs through the subsequence $\{\varepsilon_n\}$. If e is any edge with $R^0(e) = R(e) > 0$ and its endpoints v and w in $\mathcal{C}_1$, then

$$I^{\varepsilon_n}(v,e) = \frac{V^{\varepsilon_n}(w)-V^{\varepsilon_n}(v)}{R^{\varepsilon_n}(e)}$$

(see (11.40)) also converges to some $\tilde{I}(v,e)$ and

$$\tilde{V}(w)-\tilde{V}(v) = R(e)\tilde{I}(v,e).$$

The main difficulty is to show that

$$\tilde{V}(w(e,v)) = \tilde{V}(v) \quad \text{if} \quad R(e) = 0 \text{ and } v \in \mathcal{C}_1 . \tag{11.77}$$

This does not follow from the above arguments but comes from the following separate argument. Let C_i be an equivalence class of vertices, i.e., a maximal class of vertices which are connected by paths of zero resistance (see the text following (11.45)). Either $C_i \subset \mathcal{C}_1$ or C_i is disjoint from $\mathcal{C}_1$, by definition of C_i and $\mathcal{C}_1$. We are interested in the case with $C_i \subset \mathcal{C}_1$ and $\#C_i > 1$. Let $v \in C_i \setminus (A^0 \cup A^1)$ and write (11.41) for V^{ε} as follows

$$V^{\varepsilon}(v) = \frac{\frac{1}{\varepsilon}\sum_0 V^{\varepsilon}(w(e,v)) + \sum_1 \frac{V^{\varepsilon}(w(\varepsilon,v))}{R(e)+\varepsilon}}{\sum_0 \frac{1}{\varepsilon} + \sum_1 \frac{1}{R(e)+\varepsilon}} ,$$

where $\sum_0$ is the sum over edges e incident to v with $R(e) = 0$ (and hence $R^{\varepsilon}(e) = \varepsilon$) and $\sum_1$ is the sum over edges e incident to v with $R(e) > 0$. By letting ε run through the sequence $\{\varepsilon_n\}$ we obtain - whenever $\#C_i > 1$ and hence $\sum_0$ nonempty -

$$\tilde{V}(v) = \frac{\sum_0 V(w(e,v))}{\sum_0 1} , \quad v \in C_i \setminus (A^0 \cup A^1). \tag{11.78}$$

Thus on $C_i \setminus (A^0 \cup A^1)$ $\tilde{V}(v)$ is the average over its neighbors connected to v by an edge of zero resistance. Of course, all these neighbors have to belong to C_i as well, by definition of C_i. Now assume that C_i is disjoint from A^1 and that $\tilde{V}$ achieves its maximum over C_i at $v_0 \in C_i$. If $v_0 \in A^0$, then $\tilde{V}(v_0) = 0$ (by (11.38)) and hence $\tilde{V}(i) \equiv 0$ on C_i. If $v_0 \notin A^0$, then by (11.78) $\tilde{V}(w) = \tilde{V}(v_0)$ at each point w which is connected to v_0 by a path of zero resistance. Thus also in this case $\tilde{V}(\cdot)$ has the constant value $\tilde{V}(v_0)$ on C_i. The same argument with $\min_{v \in C_i} \tilde{V}(v)$ replacing $\max_{v \in C_i} \tilde{V}(v)$ works when C_i is disjoint from A^0. In the last possible case when C_i intersects A^0 and A^1 we already proved (11.76), so that this case does not have to be considered. We have therefore proved (11.77) and (11.76) follows quickly now. Indeed, (11.77) shows that $\tilde{V}$ is constant on any C_i. If we denote this constant value by $\tilde{V}(C_i)$, then one immediately sees that $\tilde{V}$ and $\tilde{I}$ must satisfy (11.46)-(11.49), at least when we restrict ourselves to $\mathcal{G}_{|\mathcal{C}_1}$ as in Lemma 11.2. These equations have the unique solution $V_{|\mathcal{C}_1}$. Thus $\tilde{V}_{|\mathcal{C}_1} = V_{|\mathcal{C}_1}$ and (see (11.50))

$$(11.79) \qquad R = \{\sum \frac{\tilde{V}(C_j)}{R(e)}\}^{-1}$$

where the sum is over all edges e of $\mathcal{G}_{|\mathcal{C}_1}$ having one endpoint in some C_i which intersects A^0 while the other endpoint lies in a C_j which is disjoint from A^0. The fact that we may restrict the sum to edges of $\mathcal{G}_{|\mathcal{C}_1}$ is in the proof of Lemma 11.2. But also, by (11.56) with $\mathcal{B} = A^0 \cup \{C_i : C_i$ intersects $A^0\}$, we have

$$(11.80) \qquad R^{\varepsilon} = \{\sum \frac{V^{\varepsilon}(w(e,v)) - V^{\varepsilon}(v)}{R^{\varepsilon}(e)}\}^{-1},$$

where the sum runs over all edges e of $\mathcal{G}_{|\mathcal{C}_1}$ with one endpoint v in some C_i which intersects A^0, and the other endpoint $w(e,v)$ in a C_j which does not intersect A^0. Again the restriction to edges of $\mathcal{G}_{|\mathcal{C}_1}$ rather than $\mathcal{G}$ makes no difference. Now let $\varepsilon \to 0$ through the sequence ε_n. $V^{\varepsilon_n}(v)$ converges to $\tilde{V}(C_i) = V(C_i) = 0$ if v belongs to a C_i which intersects A^0, and $V^{\varepsilon_n}(w) \to \tilde{V}(C_j) = V(C_j)$ if $w \in C_j$. Moreover $R(e) > 0$ for each e appearing in the right hand side of (11.80). Thus, the right hand side of (11.80) converges to the right hand side of (11.79). This proves (11.76) and the lemma. □

11.4 Proofs of Theorems 11.2 and 11.3.

We remind the reader that $B_n = [0,n]\times[0,n]$, $A_n^0 = \{0\}\times[0,n]$, $A_n^1 = \{n\}\times[0,n]$ and R_n is the resistance of $\mathbb{Z}^2 \cap B_n$ between A_n^0 and A_n^1.

Proof of Theorem 11.2. To prove (11.9) recall that, by (11.58), $R_n = 0$ as soon as there exists any path on $\mathbb{Z}^2 \cap B_n$ from A_n^0 to A_n^1 all of whose edges have zero resistance. The probability of this event is at least equal to

$$P_{p(0)}\{\exists \text{ occupied horizontal crossing on } \mathcal{G}_1 \text{ of } [0,n]\times[0,n]\}$$

$$= \sigma((n,n);\ 1,p(0),\mathcal{G}_1),$$

as one can see from the relation between bond-percolation on $\mathbb{Z}^2$ and site-percolation on its covering graph $\mathcal{G}_1$ (see Comment 2.5(iii) and Prop. 3.1). By (7.14) and the definition (3.33) this shows

$$P\{R_n = 0\} \geq 1-\sigma^*((n+2\Lambda_4,n-2\Lambda_4);2,p(0);\mathcal{G}_1)$$

$$= 1-\sigma((n+2\Lambda_4,n-2\Lambda_4);2,1-p(0),\mathcal{G}_1^*)$$

for some constant $\Lambda_4 = \Lambda_4(\mathcal{G}_1)$. Finally $p(0) > \frac{1}{2} = p_H(\mathcal{G}_1)$ is equivalent to $1-p(0) < \frac{1}{2} = p_H(\mathcal{G}_1^*) = p_T(\mathcal{G}_1^*)$ (see Application 3.4 (ii)). Thus by Theorem 5.1 (see also the end of proof of Lemma 5.4)

$$\sigma(n+2\Lambda_4,n-2\Lambda_4,2,1-p(0),\mathcal{G}_1^*) \leq C_1(2n+4\Lambda_4)e^{-C_2 n}$$

for $p(0) > \frac{1}{2}$. (11.9) follows from these estimates and the Borel-Cantelli lemma (see Renyi (1970) Lemma VII.5A).

The proof of (11.10) is very similar. By (11.59) $R_n = \infty$ whenever there does not exist a conducting path on $\mathbb{Z}^2 \cap B_n$ from A_n^0 to A_n^1. Again by the relation between bond-percolation on $\mathbb{Z}^2$ and site-percolation on $\mathcal{G}_1$, as in Comment 2.5(iii) we get from this

$$P\{R_n = \infty\} \geq 1-\sigma((n-2,n+2));\ 1,1-p(\infty),\mathcal{G}_1).$$

For $1-p(\infty) < \frac{1}{2} = p_H(\mathcal{G}_1) = p_T(\mathcal{G}_1)$ we again get from Theorem 5.1

$$P\{R_n = \infty\} \geq 1-C_1 e^{-C_2 n} .$$

An application of the Borel-Cantelli lemma now proves (11.10).

For the upper bound in (11.11) we shall use Theorem 11.1 and Lemmas 11.1 and 11.4. Note that the proof below works just as well if B_n is replaced by $[0,n] \times [0,n-m]$ and A_n^0 by $\{0\} \times [0,n-m]$, A_n^1 by $\{n\} \times [0,n-m]$ for any fixed integer m. This will be relevant for the proof of the lower bound in (11.11) later on. Now let M be some large integer and assume we can find k paths $r^1,\dots,r^k$ in $[0,n] \times [M,n-M] \subset B_n$ from A_n^0 to A_n^1 such that $R(e) < \infty$ for each edge e appearing in any of these paths, and such that r^i and r^j have no edge in common. By Lemma 11.1 we do not change R_n if we replace $R(e)$ by

$$\tilde{R}(e) := \begin{cases} 0 & \text{if } \mathring{e} \text{ is surrounded by a circuit } r \text{ or } \\ & \mathbb{Z}^2 \text{ which lies in } B_n \text{ and all of whose} \\ & \text{edges have zero resistance,} \\ R(e) & \text{otherwise .} \end{cases}$$

After this replacement, $\tilde{R}^j$:= resistance of the path r^j equals

$$\sum_i \tilde{R}(e_i^j),$$

where $e_1^j, e_2^j, \dots$ are the successive edges in r^j (see the lines preceding Lemma 11.4). But the paths $r^1,\dots,r^k$ are almost parallel resistances between A_n^0 and A_n^1. They can fail to be parallel because the paths can intersect in vertices. If two edges e_i^j and e_k^ℓ have an endpoint v in common we can think of this as a link of zero resistance between an endpoint of e_i^j and e_k^ℓ. Removing the link is equivalent to giving it infinite resistance. By Lemma 11.4 this removal can only increase the resistance of the network. Thus R_n has at most the value of the resistance of the network consisting of k parallel resistances $r^1,\dots,r^k$, i.e.,

$$R_n \le \left\{\frac{1}{\tilde{R}^1} + \dots + \frac{1}{\tilde{R}^k}\right\}^{-1} .$$

Since the harmonic mean of positive quantities is no more than their arithmetic mean (by Jensen's inequality; cf. Rudin (1966), Theorem 3.3), and since all e_i^j are distinct and contained in $[0,n] \times [M,n-M]$ and have finite resistance (by our choice of $r^1,\dots,r^k$), we obtain

$$(11.81) \qquad R_n \le \frac{1}{k^2} \sum_{j=1}^{k} \tilde{R}^j = \frac{1}{k^2} \sum_{j=1}^{k} \sum_i \tilde{R}(e_i^j)$$

$$\leq \frac{1}{k^2} \sum_{e \subset [0,n] \times [M,n-M]} \tilde{R}(e) I[R(e) < \infty] .$$

Finally, denote by $m(e) = (m_1(e), m_2(e))$ the midpoint of the edge e, and by $I(e) = I_M(e)$ the indicator function of the event

$\{\overset{\circ}{e}$ does not lie inside any circuit on $\mathbb{Z}^2$ made up of edges with zero resistance and contained in the square $[m_1(e)-M+1, m_1(e)+M-1] \times [m_2(e)-M+1, m_2(e)+M-1]\}$.

Then

$$\tilde{R}(e) I[R(e) < \infty] \leq R(e) I[R(e) < \infty] I_M(e) \qquad \text{for } e \subset [M,n-M] \times [M,n-M], \tag{11.82}$$

$$\tilde{R}(e) I[R(e) < \infty] \leq R(e) I[R(e) < \infty] \quad \text{for all } e \subset B_n . \tag{11.83}$$

By the ergodic theorem (Tempel'man (1972), Theorem 6.1, Cor. 6.2 or Dunford and Schwartz (1958), Theorem VIII. 6.9; also use Harris (1960), Lemma 3.1 and the fact that $R(e) \geq 0$)

$$\lim_{n\to\infty} \frac{1}{2n^2} \sum_{e \subset [M,n-M]^2} R(e) I[R(e) < \infty] I_M(e) = E\{R(e_0) I[R(e_0) < \infty] I_M(e_0)\} \tag{11.84}$$

with probability one (e_0 is an arbitrary fixed edge). Furthermore, if

$$\int_{(0,\infty)} x \, dF(x) = E\{R(e_0) I[R(e_0) < \infty]\} < \infty \tag{11.85}$$

then by Birkhoff's ergodic theorem (Walters (1982), Theorem 1.14)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{e \subset [0,M] \times [0,n]} R(e) I[R(e) < \infty] = 2(M+1) E\{R(e_0) I[R(e_0) < \infty]\} < \infty$$

and consequently

$$\lim_{n\to\infty} \frac{1}{2n^2} \sum_{e \subset [0,M] \times [0,n]} R(e) I[R(e) < \infty] = 0 \tag{11.86}$$

with probability one. Also, under (11.85) the ergodic theorem implies

(11.87) $$\frac{1}{2n^2}\sum_{e\subset[n-M,n]\times[0,n]}\{R(e)I[R(e)<\infty]\}$$

$$= \frac{1}{2n^2}\sum_{e\subset B_n}\{R(e)I[R(e)<\infty]\}$$

$$- \frac{1}{2n^2}\sum_{\overset{\circ}{e}\subset[0,n-M)\times[0,n]}\{R(e)I[R(e)<\infty]\}$$

$$\to \int_{(0,\infty)} x\, dF(x) - \int_{(0,\infty)} x\, dF(x) = 0 \quad (n\to\infty)$$

with probability one. Of course, we may assume that (11.85) holds, since the upper bound in (11.11) is vacuous otherwise. It follows from (11.82)-(11.87) that

$$\limsup_{n\to\infty}\frac{1}{2n^2}\sum_{e\subset[0,n]\times[M,n-M]}\tilde{R}(e)I[R(e)<\infty]$$

$$\le E\{R(e_0)I[R(e_0<\infty)]I_M(e_0)\} .$$

Together with (11.81) this implies for each fixed M

(11.88) $$\limsup R_n \le 2\limsup\left(\frac{n}{k(n,M)}\right)^2 E\{R(e_0)I[R(e_0)<\infty]I_M(e_0)\},$$

where

$k(n,M)$ = maximal number of edge-disjoint conducting paths in $[0,n]\times[M,n-M]$ from A_n^0 to A_n^1 .

(We call $r^1,\dots,r^k$ edge-disjoint if r^i and r^j have no common edges for $i\neq j$.) A simple translation of (11.4) from site-percolation on $\mathcal{G}_1$ to bond percolation on $\mathbb{Z}^2$ as in Comment 2.5(iii) gives for large n

$$P\{k(n,M)\ge\frac{1}{2}C_1(\tfrac{1}{2}-p(\infty))^{\delta_1}n\}\ge 1-2C_2 n\exp -C_3(\tfrac{1}{2}-p(\infty))^{\delta_2}n .$$

(Recall that $p_H(\mathcal{G}_1)=\frac{1}{2}$ by Application 3.4(ii) and that $P\{R(e)<\infty\} = 1-p(\infty)$.) Thus, by the Borel-Cantelli lemma

$$\limsup\left(\frac{n}{k(n,M)}\right)\le\frac{2}{C_1}(\tfrac{1}{2}-p(\infty))^{-\delta_1}$$

with probability one, for each M. In view of (11.88) this gives

$$\limsup R_n \le \frac{8}{C_1^2}(\tfrac{1}{2}-p(\infty))^{-2\delta_1}\lim_{M\to\infty}E\{R(e_0)I[R(e_0)<\infty]I_M(e_0)\}$$

with probability one. We complete the proof of the upper bound in (11.11) by showing that

$$(11.89)\qquad \lim_{M\to\infty} E\{R(e_0)I[R(e_0) < \infty]I_M(e_0)\} = \frac{1}{1-p(0)}\,\theta_2(1-p(0))\int_{(0,\infty)} x dF(x).$$

To prove (11.89) we observe that

$$(11.90)\qquad I_M(e_0) \downarrow I[e_0 \text{ is not surrounded by a circuit on } \mathbb{Z}^2 \text{ made up of edges with zero resistance}],\ M\uparrow\infty.$$

Now consider the following bond-percolation problem on $(\mathbb{Z}^2)_d$. Call and edge f^* of $(\mathbb{Z}^2)_d$ open if the edge f of $\mathbb{Z}^2$ which intersects f^* has non-zero resistance, and blocked otherwise. Then, if the open cluster of e_0^* on $(\mathbb{Z}^2)_d$ is non-empty and finite, it must be contained inside a circuit on $\mathbb{Z}^2$ made up of zero resistances. This follows from Whitney's Theorem (Whitney 1933), Theorem 4) as explained in Harris (1960), Lemma 7.1 and Appendix 2. Compare also Example 1 in Hammersley (1959). We proved an analogue for site-percolation in Cor. 2.2. and the above result can be obtained from Cor. 2.2. by the usual translation from bond-percolation on $(\mathbb{Z}^2)_d$ (which is isomorphic to $\mathbb{Z}^2$) to site-percolation on the covering graph $\mathcal{G}_1$ of $\mathbb{Z}^2$ (see Comment 2.5 (iii) and Prop. 3.1). The open cluster of e_0^* is non-empty iff e_0^* is open, or equivalently iff $R(e_0) > 0$. Moreover, the probability that any edge is open is $1-p(0)$. It follows from these observations, that the expectation of the limit of $I_M(e_0)$ in (11.90) is just

$$P\{\text{open cluster of } e_0^* \text{ on } (\mathbb{Z}^2)_d \text{ is infinite} \mid e_0^* \text{ is open}\} = \frac{\theta_2(1-p(0))}{1-p(0)}.$$

(11.89) is immediate from this since $R(e_0)$ is independent of $I_M(e_0)$. This completes the proof of the upper bound in (11.11).

The lower bound in (11.11) can be proved by a direct argument which does not rely on $\mathbb{Z}^2$ being self-dual (see Remark 11.4(i) below for an indication of such a proof). Here we shall merely appeal to the fact that by Lemma 11.3

$$\frac{1}{R_n} = S([\tfrac{1}{2}, n-\tfrac{1}{2}] \times [-\tfrac{1}{2}, n+\tfrac{1}{2}]).$$

However, $(\mathbb{Z}^2)_d$ is isomorphic to $\mathbb{Z}^2$ so that $S([\frac{1}{2}, n-\frac{1}{2}] \times [-\frac{1}{2}, n+\frac{1}{2}])$

has the same distribution as $R([0,n+1]\times[0,n-1])$ when the distribution of an individual edge is given by

$$(11.91)\qquad \begin{aligned} &P\{R(e) = 0\} = p(\infty), \\ &P\{R(e) = \infty\} = p(0), \\ &P\{R(e) \varepsilon B\} = \int_{\frac{1}{x}\varepsilon B} dF(x),\ B \subset (0,\infty). \end{aligned}$$

(Compare (11.6)-(11.8) and recall that $S(e^*) = 1/R(e)$ in Lemma 11.3.) The lower bound in (11.11) now follows by applying the upper bound in (11.11) to $R([0,n+1]\times[0,n-1])$ when the distribution of $R(e)$ is as given by (11.91) instead of (11.6)-(11.8). (Note that the upper bound applies just as well to $R([0,n+1]\times[0,n-1])$ as to $R_n = R([0,n]\times[0,n])$ as pointed out in the beginning of the proof of (11.11).) □

Corollary 11.1 is immediate from (11.11) and (8.4).

Proof of Theorem 11.3. We do not give a detailed proof of (11.22). Its proof is a simplified version of the proof of the upper bound in (11.11). This time we do not use Lemma 11.1 and do not replace $R(e)$ by $\tilde{R}(e)$. We simply find enough edge disjoint conducting paths from A_n^0 to A_n^1 by applying Theorem 11.1 to the restrictions of B_n to planes specified by fixing $x(3),\dots,x(d)$, i.e., to graphs which are the restrictions of $\mathbb{Z}^d$ to $[0,n]\times[0,n]\times\{i(3)\}\times\dots\times\{i(d)\}$, with $0 \leq i(3) \leq n,\dots,$ $0 \leq i(d) \leq n$.

(11.19) is quite trivial. As in (11.9) $R_n = 0$ as soon as there is a path in B_n from A_n^0 to A_n^1 all of whose edges have zero resistance. But the probability that such a connection exists in $[0,n]$ $\times [0,n]\times\{i(3)\}\times\dots\times\{i(d)\}$ equals the sponge-crossing probability $S_{p(0)}(n,n)$ of Seymour and Welsh (1978). By their results (see pp.233, 234) for $p(0) \geq \frac{1}{2}$

$$S_{p(0)}(n,n) \geq S_{1/2}(n,n) \geq S_{1/2}(n,n+1) = \frac{1}{2}^{1)}.$$

Since crossings in $[0,n]\times[0,n]\times\{i(3)\}\times\dots\times\{i(d)\}$ for different $i(3),\dots,i(d)$ are independent, it follows that

---

1) Here we merely need that $S_{1/2}(n,n)$ is bounded away from zero. By going over to site-percolation we can also obtain this from Theorem 5.1. However, the proof of Seymour and Welsh (1978) is much simpler in the special case of bond-percolation on $\mathbb{Z}^2$.

$$P\{R_n = 0\} \geq 1-2^{-(n+1)^{d-2}} \quad \text{if} \quad p(0) \geq \frac{1}{2}.$$

(11.19) thus follows from the Borel-Cantelli lemma.

Also (11.20) is easy. By Theorem 5.1, (5.16) and the end of the proof of Lemma 5.4 (see (5.55)) one has for $1-p(\infty) < p_{S,d}$

$$P\{\exists \text{ conducting path in } B_n \text{ from } A_n^0 \text{ to } A_n^1\}$$

$$\leq \sum_{v \varepsilon A_n^0} P\{\text{number of edges reachable by a conducting path from } v \text{ is at least } n\}$$

$$\leq (n+1)^{d-1} C_1 e^{-C_2 n} .$$

Again, it follows from the Borel-Cantelli lemma that with probability one for all large n there does not exist a conducting path in B_n from A_n^0 to A_n^1. In view of (11.59) this implies (11.20).

We finally turn to (11.21). Its proof rests on Lemma 11.2. First we replace the resistance of each edge e which does not have each end-point in $\mathcal{C}_1$ by ∞. Here $\mathcal{C}_1$ is as in Lemma 11.2 with $\mathcal{G}$ = restriction of $\mathbb{Z}^d$ to B_n. This replacement does not change R_n. Denote the mid-point of e by $m(e) = (m_1(e),\dots,m_d(e))$ and set

> $J_M(e) = 0$ if there exists a conducting path in the full network $\mathbb{Z}^d$ from one of the endpoints of e to one of the two hyperplanes $x(1) = m_1(e) \pm M$,

and $J_M(e) = 1$ otherwise. Then for $M < m_1(e) < n-M$, $J_M(e) = 1$ implies that both endpoints of e are outisde $\mathcal{C}_1$, since they are not connected to $A_n^0 \cup A_n^1$ by a conducting path in $\mathbb{Z}^d \cap B_n$. Therefore the modified resistance for such edges is at least $R(e) + J_M(e)\cdot\infty$. Thus by Lemma 11. R_n is at least as large as the resistance between A_n^0 and A_n^1 when $R(e)+J_M(e)\cdot\infty$ is used instead of $R(e)$ for the resistance of each edge e with $M < m_1(e) < n-M$. We next reduce to zero the resistances of all "vertical" edges between two neighbors $(i(1),\dots,i(d))$ and $(i(1),i(2),\dots,i'(s),\dots,i(d))$ with $2 \leq s \leq d$, $|i'(s)-i(s)| = 1$. By Lemma 11.4 once more this does not increase R_n. Set

$$\tilde{R}(e) = \begin{cases} R(e)+J_M(e)\cdot\infty & \text{if } e \text{ is a "horizontal" edge and } M < m_1(e) < n-M \\ R(e) & \text{if } e \text{ is a "horizontal" edge with } 0 \leq m_1(e) \leq M \text{ or } n-M \leq m_1(e) \leq n \\ 0 & \text{if } e \text{ is a "vertical" edge.} \end{cases}$$

A "vertical" edge was defined above, and a horizontal edge is an edge from $(i(1),\dots,i(d))$ to $(i(1)+1,i(2),\dots,i(d))$ for some $0 \le i(1) < n$, $0 \le i(2),\dots,i(d) \le n$. Denote by $\tilde{R}_n$ the resistance between A_n^0 and A_n^1 in B_n, when $\tilde{R}(e)$ is the resistance of the generic edge e. Then, by the above $R_n \ge \tilde{R}_n$. However, $\tilde{R}_n$ is easy to calculate. As discussed in Sect. 11.3 all vertices in a "vertical plate" $\{i(1)\} \times [0,n]^{d-1}$ will have the same potential since they are connected by zero resistances. We may therefore identify these vertices to one vertex. If we do this for each $i(1) \in [0,n]$, then we obtain a graph $\mathcal{K}_n$ whose vertices we denote by $\hat{0},\dots,\hat{n}$, with $\hat{i}$ corresponding to the plate $\{i\} \times [0,n]^{d-1}$ in $\mathbb{Z}^d \cap B_n$. There are $(n+1)^{d-1}$ edges between $\hat{i}$ and $(\widehat{i+1})$ with resistances $\tilde{R}(e_{ij})$, where j runs through the $(n+1)^{d-1}$ possible values for $(i(2),\dots,i(d))$ and e_{ij} denotes the edge from (i,j) to $(i+1,j)$ (see Fig. 11.4). These resistances are

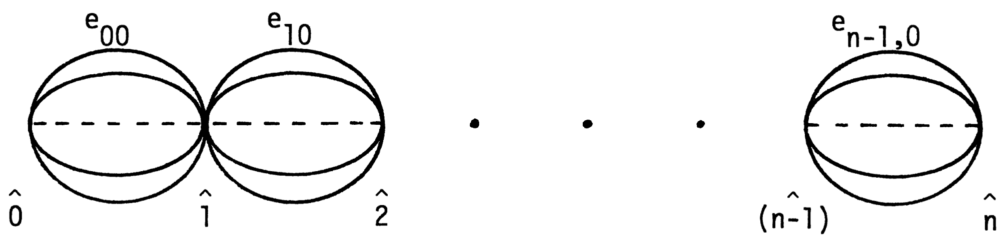

Figure 11.4. The graph $\mathcal{K}_n$ obtained by identifying vertices on "vertical plates".

in parallel, and equivalent to a single resistance of size

$$\{\sum_j (\tilde{R}(e_{ij}))^{-1}\}^{-1}$$

between $\hat{i}$ and $(\widehat{i+1})$. The resistances between $\hat{i}$ and $(\widehat{i+1})$ for $i = 0,1,\dots,n-1$ are in series so that

$$\tilde{R}_n = \sum_{i=0}^{n-1} \{\sum_j (\tilde{R}(e_{ij}))^{-1}\}^{-1} \ge \sum_{i=M}^{n-M} \{\sum_j (\tilde{R}(e_{ij}))^{-1}\}^{-1}$$

$$\ge (n-M+1)^2 \{\sum_{i=M}^{n-M} \sum_j (\tilde{R}(e_{ij}))^{-1}\}^{-1} .$$

The last inequality again results from the fact that the arithmetic mean is at least as large as the harmonic mean. Consequently, by the ergodic theorem (Tempel'man (1972), Theorem 6.1, Cor. 6.2 or Dunford and Schwartz (1958), Theorem VIII.6.9).

$$\liminf n^{d-2}R_n \geq \liminf n^{d-2}\tilde{R}_n \geq \lim n^d \{\sum_{i=M}^{n-M} \sum_j (\tilde{R}(e_{ij}))^{-1}\}^{-1}$$
$$= (E\{(\tilde{R}(e_0))^{-1}\})^{-1}$$
$$= (P\{J_M(e_0) = 0\})^{-1}\{\int \frac{1}{x}\, dF(x)\}^{-1} ,$$

for any fixed M, and any fixed edge e. Finally, as $M \to \infty$, $P\{J_M(e_0) = 0\}$ converges to

P{the cluster of all edges connected to e_0 by a conducting path is unbounded$|R(e_0) < \infty$}

$$= \frac{\theta_d(1-p(\infty))}{1-p(\infty)} .$$

This proves (11.21). □

Remark.

(i) To prove the lower bound in (11.11) without using Straley's duality lemma (Lemma 11.3) one can proceed along the lines of the above proof of (11.21). First we replace $R(e)$ by $\tilde{R}(e)$. However, we do not form $\mathcal{K}_n$ by identifying the vertices in each segment $\{i\} \times [0,n]$ now. Instead, consider disjoint vertical crossings r^{j*}, $1 \leq j \leq k$, of $[\frac{1}{2}, n - \frac{1}{2}] \times [-\frac{1}{2}, n + \frac{1}{2}]$ on $(\mathbb{Z}^2)_d$ such that all edges in each r_i^* intersect an edge of $\mathbb{Z}^2$ with strictly positive resistance. If r^{j*} contains the edges e_i^{j*}, and e_i^j is the edge of $\mathbb{Z}^2$ which intersects e_i^{j*}, then we form a vertex of $\mathcal{K}_n$ by identifying the endpoints of the e_i^j, $i = 1,2,\ldots$ which are immediately to the left of r^{j*}. Another vertex of $\mathcal{K}_n$ is formed by identifying the endpoints of the e_i^j, $i = 1,2,\ldots$ immediately to the right of r^{j*}. By choice of r^{j*}, $R(e_i^j) > 0$. After constructing $\mathcal{K}_n$ by making these identifications for each j, $1 \leq j \leq k$ we can essentially copy the rest of the proof of (11.21). All we need is a lower bound for the number k of disjoint vertical crossings r^{j*} of the above type. A lower bound of order $\frac{1}{2} C_1(\frac{1}{2} - p(0))^{\delta_1} n$ can be obtained from Theorem 11.1 in the same way as in the estimate for $k(n,M)$ in the proof of the upper bound in (11.11).