APPENDIX. SOME RESULTS FOR PLANAR GRAPHS.

In this appendix we prove several graph theoretical, or point-set topological results, in particular Propositions 2.1-2.3 and Corollary 2.2 which were already stated in Ch.2. The proofs require somewhat messy arguments, even though most of these results are quite intuitive. We base most of our proofs on the Jordan curve theorem (Newman,(1951), Theorem V. 10.2). Some more direct and more combinatorial proofs can very likely be given; see the approach of Whitney (1932, 1933). Especially Whitney (1933), Theorem 4, is closely related to Cor. 2.2., Prop. 2.2 and Prop. A.1, and has been used repeatedly in percolation theory.

Throughout this appendix $\mathcal{M}$ is a mosaic, $\mathcal{F}$ a subset of the collection of faces of $\mathcal{M}$ and $(\mathcal{G},\mathcal{G}^*)$ a matching pair based on $(\mathcal{M},\mathcal{F})$. These terms were defined in Sect. 2.2. $\mathcal{G}_{p\ell},\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$ will be the planar modifications as defined in Sect. 2.3. We fix an occupancy configuration ω on $\mathcal{M}$ and extend it as in (2.15), (2.16). $W(v)$ and $W_{p\ell}(v)$ are the occupied cluster of v on $\mathcal{G}$ and $\mathcal{M}_{p\ell}$ (or $\mathcal{G}_{p\ell}$), respectively, in the configuration ω. ∂W, the boundary of W, is defined in Def. 2.8; $v\ \mathcal{G}\ w$ means that v and w are adjacent vertices on $\mathcal{G}$.

Proposition 2.1. Let $\partial W_{p\ell}(v)$ be the boundary of $W_{p\ell}(v)$ on $\mathcal{M}_{p\ell}$. If $W_{p\ell}(v)$ is non-empty and bounded and (2.3)-(2.5) hold with $\mathcal{G}$ replaced by $\mathcal{M}$, then there exists a vacant circuit $J_{p\ell}$ on $\mathcal{M}_{p\ell}$ surrounding $W_{p\ell}(v)$, and such that all vertices of $\mathcal{M}_{p\ell}$ on $J_{p\ell}$ belong to $\partial W_{p\ell}(v)$.

We owe the idea of the proof to follow to R. Durrett. We shall write $W_{p\ell}$ and $\partial W_{p\ell}$ instead of $W_{p\ell}(v)$ and $\partial W_{p\ell}(v)$. On various occasions we shall use the symbol for a path to denote the set of points which belong to some edge in the path. Thus in (A.2), the left hand side is the set of points which belong to π and to $W \cup \partial W_{p\ell}$. In (A.5) $\text{int}(J) \setminus \tilde{\pi}$ is the set of points in $\text{int}(J)$ which do not lie on $\tilde{\pi}$. This abuse of notation is not likely to lead to confusion.

We shall actually prove the slightly stronger statement that the vertices of $\mathcal{M}_{p\ell}$ on $J_{p\ell}$ belong to $\partial_{ext} W_{p\ell}$, the "exterior boundary of $W_{p\ell}$", where

(A.1) $\partial_{ext} W_{p\ell} := \{u \in \partial W_{p\ell}: \exists$ path π from u to ∞ on $\mathcal{M}_{p\ell}$ such that u is the only point of π in $W_{p\ell} \cup \partial W_{p\ell}\}$.

The crucial property of $\partial_{ext} W_{p\ell}$ is given in the following lemma.

Lemma A.1. Assume that (2.3)-(2.5) hold with $\mathcal{G}$ replaced by $\mathcal{M}$. If $W_{p\ell}$ is non-empty and bounded, then $\partial_{ext} W_{p\ell} \neq \emptyset$. Let $u \in \partial_{ext} W_{p\ell}$, $w \in W_{p\ell}$ and π a path from u to ∞ on $\mathcal{M}_{p\ell}$ such that

$$u \, \mathcal{M}_{p\ell} \, w$$

and[1)]

(A.2) $$\pi \cap \{W \cup \partial W_{p\ell}\} = \{u\} .$$

Let e be an edge of $\mathcal{M}_{p\ell}$ from w to u and $\tilde{\pi}$ the simple path consisting of e followed by π . Then there exists a Jordan curve J in $\mathbb{R}^2$ such that

(A.3) $$u \in \text{int}(J) ,$$

(A.4) J intersects each edge of $\mathcal{M}_{p\ell}$ incident to u exactly once, but all edges of $\mathcal{M}_{p\ell}$ not incident to u belong to ext(J),

(A.5) $\text{int}(J)\backslash\tilde{\pi}$ has exactly two components, K' and K" say. Any edge between u and a vertex $\tilde{u} \in \partial_{ext} W_{p\ell}$ intersects exactly one of the components K' and K". There exists a vertex $u' \in \partial_{ext} W_{p\ell}$ and an edge e' of $\mathcal{M}_{p\ell}$ between u and u' which intersects only K'. There also exists a vertex $u'' \in \partial_{ext} W_{p\ell}$ and an edge e'' of $\mathcal{M}_{p\ell}$ between u and u'' which intersects only K" ($u' = u''$ is possible!).

(Fig. A.1 gives a schematic illustration of the situation.)

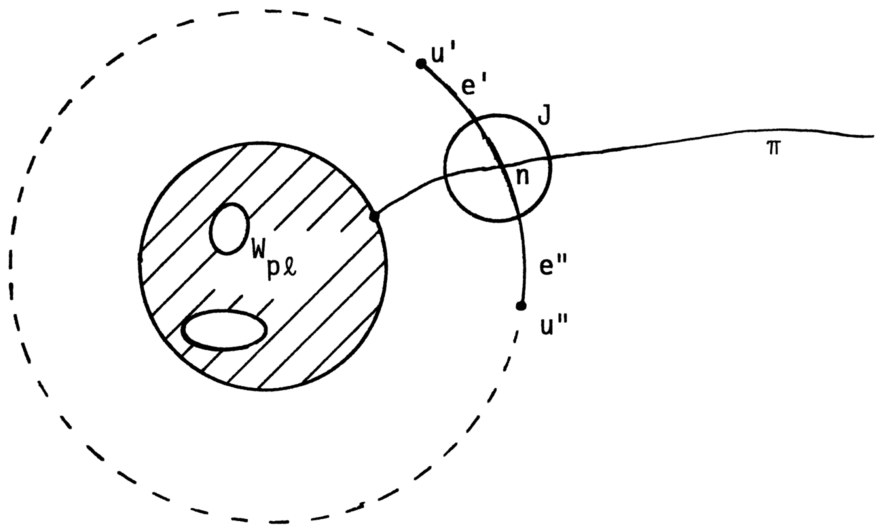

Figure A.1. $W_{p\ell}$ is the hatched region. The vertices of $\partial_{ext}W$ are on the dashed curve. J is the small circle surrounding u.

Proof: If $W_{p\ell}$ is non-empty and bounded, then any path from ∞ to $W_{p\ell}$ must intersect $\partial W_{p\ell}$ a first time. This intersection belongs to $\partial_{ext} W_{p\ell}$. Thus $\partial_{ext} W_{p\ell} \neq \emptyset$ in this situation.

Now take $u \in \partial_{ext} W_{p\ell}$. By definition there exists a simple path π from u to ∞ on $\mathcal{M}_{p\ell}$ satisfying (A.2) and a $w \in W_{p\ell}$ which is adjacent to u. The self-avoiding path π cannot intersect e in its interior (because $\mathcal{M}_{p\ell}$ is planar), nor in the point w (by (A.2)), and goes through the point u only once (at its beginning). Thus $\tilde{\pi}$ has no double points. Now let D be a small open disc around u such that $\bar{D}$ does not intersect any edge of $\mathcal{M}_{p\ell}$ not incident to u. (Use (2.4) to find such a disc). If all edges incident to u are piecewise linear, then the perimeter of D will satisfy (A.3) and (A.4) provided D is sufficiently small. The general situation can be reduced to this simple case by means of a homeomorphism of $\mathbb{R}^2$ onto itself which takes pieces of the edges of $\mathcal{M}_{p\ell}$ incident to u onto straight line segments radiating from the origin (see Newman (1951), exercise VI. 18.3 for the existence of such a homeomorphism). We may therefore assume that we have a Jordan curve J satisfying (A.3) and (A.4).

Note that e, as well as the unique edge of π incident to u (the first edge of π) each intersect J exactly once (by (A.4)) so that $\tilde{\pi}$ intersects J exactly twice, and $\text{int}(J)\backslash\tilde{\pi}$ has indeed two components - which we call K' and K'' (see Newman (1951), Theorem V. 11.7). Let $e_0 = e$, and let $e_1, e_2, \ldots, e_{\nu-1}$, $e_\nu = e_0$ be the edges of $\mathcal{M}_{p\ell}$ incident to u, listed in the order in which they intersect J as we traverse J in one direction from $e_0 \cap J$; there are only finitely many of these by (2.4). Write u_i for the endpoint of e_i different from u, and x_i for the intersection of e_i and

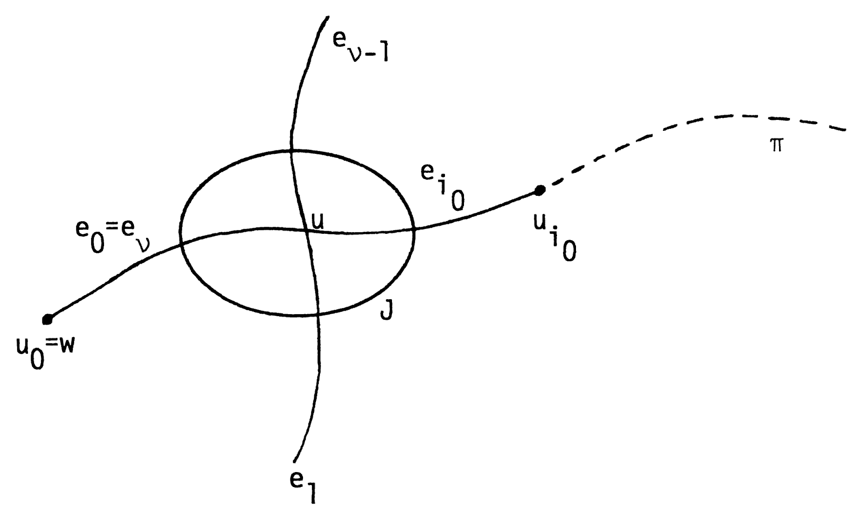

Figure A.2.

J. Thus $u_0 = w$. The first edge of π is one of the e_i, say e_{i_0}. For $i \neq 0, i_0, \nu$, e_i runs from u to x_i inside one component K' or K'', and then from x_i to u_i it is in ext(J) by (A.4) (note $u_i \in \text{ext}(J)$, also by (A.4)). Thus, each of these edges intersect exactly one of K' and K''. Since each of the two arcs of J from x_0 to x_{i_0} form part of the boundary of one the components K' and K'' (Newman (1951), Theorem V.11.8), it follows that e_i, $1 \leq i < i_0$, intersect the same component, K' say, while e_i, $i_0 < i < \nu$ intersect the other, which will be K''. This proves the first statement in (A.5) (since $u_0 = u_\nu \in W$ and hence not in $\partial W_{p\ell}$

and also $u_{i_0} \varepsilon \pi$ does not belong to $\partial W_{p\ell}$).

We write A_i for the arc of J from x_i to x_{i+1}, $0 \leq i \leq \nu-1$. Then $A_i \setminus \{x_i, x_{i+1}\}$ does not intersect any edge and therefore lies entirely in one face of $\mathcal{M}_{p\ell}$. Since all faces of $\mathcal{M}_{p\ell}$ are triangles (Comment 2.3(vi)), this implies that e_i and e_{i+1} lie in the boundary of a triangle, and $u_i \mathcal{M}_{p\ell} u_{i+1}$, $u_0 = w \varepsilon W_{p\ell}$, while $u_{i_0} \varepsilon \pi$ is not in $W_{p\ell}$. Hence the index

$$i_1 = \max \{j: 0 \leq j \leq i_0, u_j \varepsilon W_{p\ell}\}$$

is well defined. As observed above, u_{i_1+1} is a neighbor of $u_{i_1} \varepsilon W_{p\ell}$, but by definition of i, $u_{i_1+1} \notin W_{p\ell}$. Therefore, $u_{i_1+1} \varepsilon \partial W_{p\ell}$. Also, $u_{i_0} \varepsilon \pi$ does not belong to $\partial W_{p\ell}$ by (A.2). Thus $i_1+1 < i_0$ and we can define i_2 by

$$i_2 = \max \{j: i_1 < j < i_0, u_j \varepsilon \partial W_{p\ell}\} .$$

We can connect u_{i_2} to ∞ by a path consisting of edges from u_j to u_{j+1}, $i_2 \leq j < i_0$, followed by the piece of π from u_{i_0} to ∞. The vertices $u_{i_2+1}, \ldots, u_{i_0}$ do not belong to $W_{p\ell} \cup \partial W_{p\ell}$ by choice of i_1, i_2, so that $u_{i_2} \varepsilon \partial_{ext} W_{p\ell}$ with $0 < i_2 < i_0$. Finally we define

$$i' = \min\{0 < j \leq i_2 : u_j \varepsilon \partial_{ext} W_{p\ell}\} .$$

By the above i' is well defined, and $u' := u_{i'}$ is connected to u by an the edge $e_{i'}$ which intersects K', but does not intersect K''. Similarly we can define

$$i'' = \max\{i_0 < j < \nu : u_j \varepsilon \partial_{ext} W_{p\ell}\}$$

and $u'' = u_{i''}.e_{i''}$ only intersects K''. This proves the existence of the desired u', e', u'' and e'' for A.5. □

Proof of Proposition 2.1: For a non-empty and bounded $W_{p\ell}$ pick any $u_0 \varepsilon \partial_{ext} W_{p\ell}$ and apply Lemma 1 with u_0 for u. Let u_1 be one of the vertices u', $u'' \varepsilon \partial_{ext} W_{p\ell}$ adjacent to u_0 whose existence is guaranteed by Lemma A.1. Say we picked u' for u. Let e_1

be an edge between $u_0 = u$ and $u_1 = u'$ which intersects only K' as in Lemma A.1. Assume we have already constructed $u_0, e_1, u_1, \ldots, e_i, u_i$ with $u_i \in \partial_{ext} \mathcal{M}_{p\ell}$ and e_j an edge of $\mathcal{M}_{p\ell}$ between u_{j-1} and $u_j, e_{j-1} \neq e_j$, $1 \leq j \leq i$. We then apply Lemma A.1 to u_i. Associated with u_i are two components K' and K''. Assume e_i intersects K'. Then by (A.5) we can find an edge e_{i+1} from u_i to some $u_{i+1} \in \partial_{ext} W_{p\ell}$, such that e_{i+1} intersects only K'' and not K', and hence with $e_{i+1} \neq e_i$. We continue in this way until the first time we obtain a double point, i.e., to the smallest index ν for which there exists a $\rho < \nu$ with $u_\rho = u_\nu$. $\nu < \infty$ because $W_{p\ell}$ is bounded, and therefore $\partial_{ext} W_{p\ell} \subset \partial W_{p\ell}$ finite (see (2.3), (2.4)). ρ will be unique by the minimality of ν. Since $\mathcal{M}_{p\ell}$ is planar, $J_{p\ell} = (u_\rho, e_\rho, \ldots, e_\nu, v_\nu)$ - or more precisely the curve made up from $e_\rho, e_{\rho+1}, \ldots, e_\nu$ - is a Jordan curve. We now show that it has the required properties. The vertices on $J_{p\ell}$ belong to $\partial_{ext} W_{p\ell} \subset \partial W_{p\ell}$ by choice of the u_i, and since each vertex of $\partial W_{p\ell}$ has to be vacant, $J_{p\ell}$ is vacant. To show that $W_{p\ell} \subset \mathrm{int}(J_{p\ell})$ observe first that all vertices of $J_{p\ell}$ belong to $\partial W_{p\ell}$ and therefore not to $W_{p\ell}$. Thus $W_{p\ell} \cap J_{p\ell} = \emptyset$ and the connected set $W_{p\ell}$ lies entirely in one component of $\mathbb{R}^2 \setminus J_{p\ell}$. Now write u for $u_{\rho+1}$ and let π be a path on $\mathcal{M}_{p\ell}$ from u to ∞ satisfying (A.2), and e an edge of $\mathcal{M}_{p\ell}$ from u to some $w \in W_{p\ell}$. We apply Lemma A.1 once more with this choice of u, π, w and e. With $\tilde{\pi}$ and J as in Lemma A.1 we may assume (by virtue of the construction of $J_{p\ell}$) that the two edges e_ρ and $e_{\rho+1}$ incident to u intersect different components of $\mathrm{int}(J) \setminus \tilde{\pi}$. We shall prove now that this implies

(A.6) $\tilde{\pi}$ crosses $J_{p\ell}$ from $\mathrm{ext}(J_{p\ell})$ to $\mathrm{int}(J_{p\ell})$ at u.

This will suffice, since the part $\pi \setminus \{u\}$ of $\tilde{\pi}$ clearly lies in $\mathrm{ext}(J_{p\ell})$, so that (A.6) will imply that $e \setminus \{u\}$ belongs to $\mathrm{int}(J_{p\ell})$. In particular w will belong to $\mathrm{int}(J_{p\ell})$. Hence $W_{p\ell} \subset \mathrm{int}(J_{p\ell})$ and $J_{p\ell}$ surrounds $W_{p\ell}$.

To prove (A.6) note that the Jordan curve J surrounding u, constructed in Lemma A.1 intersects $J_{p\ell}$ in two points only, say x' on e_ρ and x'' on $e_{\rho+1}$ (by (A.4)). The two open arcs of J between x' and x'' must lie in different components of $\mathbb{R}^2 \setminus J_{p\ell}$, one in $\mathrm{int}(J_{p\ell})$ and the other in $\mathrm{ext}(J_{p\ell})$. Indeed each of these arcs lies entirely in one component of $\mathbb{R}^2 \setminus J_{p\ell}$, and they cannot both lie in the same component, because $u \in J_{p\ell}$ lies on the boundary of $\text{int}(J_{p\ell})$ as well as the boundary of $\text{ext}(J_{p\ell})$. Thus, there exists continuous curves from J to points in its interior near u which lie in $\text{int}(J_{p\ell})$, and there also are such curves in $\text{ext}(J_{p\ell})$. Now we have by (A.4) (or more directly by its proof) that π intersects J exactly once, in y' say, and e also intersects J exactly once, in y'' say (see Fig. A.3).

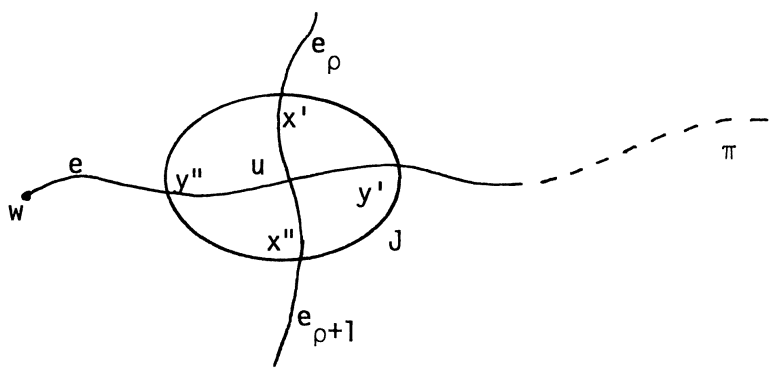

Figure A.3

x' and x'' cannot lie on the same arc of J between y' and y'' because x' and x'' are the endpoints of the pieces of $e_\rho \cap \text{int}(J)$ and $e_{\rho+1} \cap \text{int}(J)$, respectively, while by construction $e_\rho \cap \text{int}(J)$ and $e_{\rho+1} \cap \text{int}(J)$ belong to different components of $\text{int}(J) \setminus \tilde{\pi}$. These two different components, K' and K'', each have one of the arcs of J from y' to y'' in their boundary, so that x' has to lie in the arc bounding K' and x'' in the other arc, bounding K''. But this means that y' and y'' separate x' and x'' on J. Therefore, y' and y'' do not lie on the same arc of J between x' and x''. Since we saw above that one of these open arcs was in $\text{int}(J_{p\ell})$ and the other in $\text{ext}(J_{p\ell})$ it follows that one of the points y' is in $\text{int}(J_{p\ell})$ and the other in $\text{ext}(J_{p\ell})$. (A.6) now follows. □

Corollary 2.2. If $W(v)$ is non-empty and bounded and (2.3)-(2.5) hold, then there exists a vacant circuit J^* on $\mathcal{G}^*$ surrounding $W(v)$.

Proof: By Cor. 2.1 $W \subset W_{p\ell}$ and by Prop. 2.1 there exists a vacant circuit $J_{p\ell}$ on $\mathcal{M}_{p\ell}$ surrounding $W_{p\ell}$, and therefore also W. Note that $J_{p\ell}$ cannot contain any central vertex of $\mathcal{G}$ since these are all occupied (cf. (2.15)). Thus, $J_{p\ell}$ is actually a circuit on $\mathcal{G}^*_{p\ell}$. Assume it is made up from the edges $e^*_1,\dots,e^*_\nu$ of $\mathcal{G}^*_{p\ell}$, and that the endpoints of e^*_i are v^*_{i-1} and v^*_i. Then $r^* = (v^*_0,e^*_1,\dots,e^*_\nu,v^*_\nu)$ is a path on $\mathcal{G}^*_{p\ell}$ with one double point, to wit $v^*_0 = v^*_\nu$. We now apply the procedure of the proof of Lemma 2.1a, with $\mathcal{G}^*$ instead of $\mathcal{G}$, to remove the central vertices from v^*. Let $0 \le i_0 < i,\dots, < i_\rho \le \nu$ be the indices for which $v^*_{i_j}$ is not a central vertex of $\mathcal{G}^*_{p\ell}$. Then, as in Lemma 2.1a $i_0 \le 1$, $i_\rho \ge \nu-1$, and $i_{j+1} - i_j \le 2$. If $i_{j+1} = i_j+1$ so that $v^*_{i_j}$ and $v^*_{i_{j+1}}$ are adjacent on $\mathcal{G}^*$, and e^*_{j+1} is an edge of $\mathcal{G}^*$, then we do not change e^*_{j+1}. If $i_{j+1} = i_j+2$, then $v^*_{i_{j+1}}$ is the central vertex on $\mathcal{G}^*$ of some face F which is close-packed in $\mathcal{G}^*$. We then replace the piece $e^*_{i_j+1}$, $v^*_{i_j+1}$, $e^*_{i_j+2}$ of r^* by the single edge of $\mathcal{G}^*$ through F, with endpoints $v^*_{i_j}$ and $v^*_{i_j+2}$. We write $\tilde{v}^*_j$ for $v^*_{i_j}$ and $\tilde{e}^*_{j+1}$ for the edge from $\tilde{v}^*_j$ to $\tilde{v}^*_{j+1}$. We make these replacements successively. Assume for the sake of argument that $i_0 = 0$ (this can always be achieved by numbering the vertices of r^* such that it starts with a non-central vertex). Assume also that we already made all replacements between $v^*_{i_0} = v^*_0$ and $v^*_{i_k}$. We then have the sequence $\tilde{v}^*_0$, $\tilde{e}^*_0,\dots,\tilde{e}^*_k,\tilde{v}^*_k$, $e^*_{i_k+1},\dots,v^* = \tilde{v}^*_0$, and can form the curve J_k made up from $\tilde{e}^*_0,\dots,\tilde{e}^*_k$, $e^*_{i_k+1}$, $e^*_{i_k+2},\dots,e^*_\nu$ (even though this is neither a curve on $\mathcal{G}^*_{p\ell}$ nor on $\mathcal{G}^*$). Assume that J_k is a Jordan curve which contains W in its interior. We shall now show that then J_{k+1}, is also a Jordan curve which contains W in its interior. This will prove the corollary, since $J_0 = J_{p\ell}$ has these properties and J_ρ or $J_{\rho+1}$ will be a curve on $\mathcal{G}$ with the properties required of J^*. If $\tilde{e}^*_{k+1} = e^*_{i_k+1}$, then there is nothing to prove. Assume therefore $i_{k+1} = i_{k+2}$ and that $\tilde{e}^*_{k+1}$ is the edge in the closed face

$\overline{F}$ of $\mathcal{M}$ from $\tilde{v}_k^* = v_{i_k}^*$ to $\tilde{v}_{k+1}^* = v_{i_k+2}^*$, while $v_{i_k+1}^*$ is the central vertex of F. By Comment 2.3(i) the three edges $e_{i_{k+1}}^*$, $e_{i_{k+2}}^*$ and $\tilde{e}_{k+1}^*$ then form the topological boundary of a closed "triangle", T say. J_{k+1} is again a Jordan curve, because it contains only vertices of J_k, and e_j^* with $i+2 < j \le \nu$ cannot intersect the interior of the edge $\tilde{e}_{k+1}^*$ of $\mathcal{G}$. The latter statement results from Comment 2.3(i) and the fact that e_j^* does not contain the central vertex $v_{i_k+1}^*$ of F, because J_k is self-avoiding. From the facts that W consists of vertices and edges of $\mathcal{G}$ and $W \subset \text{int}(J_k)$ and from Comment 2.3(i) it follows that W cannot intersect Fr(T). Since $\mathring{T}$ contains no vertices of $\mathcal{G}$, $W \subset \mathring{T}$ is also impossible so that $W \cap T = \emptyset$. But this implies $W \subset \text{int}(J_{k+1})$ because $\text{int}(J_k) \setminus \text{int}(J_{k+1}) \subset T$, and $W \subset \text{int}(J_k)$. □

In the proof of Prop. 2.2 we shall use the next lemma, which follows from Alexander's separation lemma (Newman (1951), Ch.V.9). Actually one can deduce Prop. 2.2 from Prop. 2.1 without this lemma, but it is needed a few times later on anyway. Lemma A.2 is essentially the same as Lemma 3 in Kesten (1980a).

Lemma A.2. Let J_1 be a Jordan curve in $\mathbb{R}^2$ which consists of four closed arcs A_1, A_2, A_3, A_4 with disjoint interiors, which occur in this order when J_1 is traversed in one direction. (Some of these arcs may reduce to a single point.) Further, let J_2 be a Jordan curve in $\mathbb{R}^2$ with

$$A_1 \subset \text{int}(J_2) \quad \text{but} \quad A_3 \subset \text{ext}(J_2). \tag{A.7}$$

Then J_2 contains an arc B with one endpoint each on $\mathring{A}_2$ and $\mathring{A}_4$ and such that the interior of B is contained in $\text{int}(J_1)$.

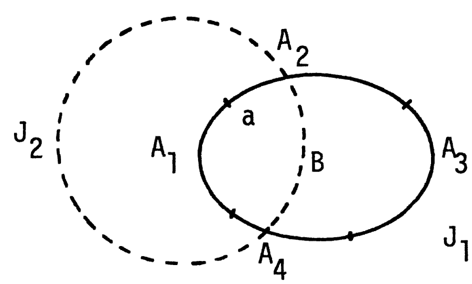

Figure A.4 J_1 is the solidly drawn curve. J_2 is dashed.

Proof: We write $\overline{J}_1$ for $J_1 \cup \mathrm{int}(J_1)$. Also for $x,y \in J_2$ and $[x,y]$ one of the closed arcs of J_2 from x to y, we shall write $(x,y]$ for $[x,y]\setminus\{x\}$ and (x,y) for $[x,y]\setminus\{x,y\}$. (x,y) is the interior of $[x,y]$. For $r = 2,4$ we define

(A.8) $G_r = \{x \in J_2 \cap \overline{J}_1$: there exists a point $y \in J_2 \cap A_r$ such that the interior (x,y) of one of the arcs of J_2 from x to y is contained in $\mathrm{int}(J_1)\}$.

The first task is to show that G_r is closed. First we observe that J_2 is closed so that

(A.9) $$\overline{G}_r \subset \text{closure of } J_2 = J_2 .$$

Now if $z \in \overline{G}_r \cap \mathrm{int}(J_1)$, then $z \in J_2 \cap \mathrm{int}(J_1)$ and it is easy to see that $z \in G_r$ in this case. We therefore restrict ourselves to showing that any $z \in \overline{G}_r \cap J_1$ lies in G_r itself. This is true by definition if $z \in J_2 \cap A_r$, since

(A.10) $$J_2 \cap A_r \subset G_r$$

(take $y = x$ in (A.8) for $x \in J_2 \cap A_r$; in this case one of the arcs from x to y has an empty interior). In addition, by virtue of (A.7),

(A.11) $$J_2 \cap (A_1 \cup A_3) = \emptyset .$$

Thus we only have to consider $z \in \overline{G}_r \cap A_4$ if $r = 2$ and $z \in \overline{G}_r \cap A_2$ if $r = 4$. For the sake of definiteness take $r = 2$, $z \in \overline{G}_2 \cap A_4$. Let $x_n \in G_2$, $x_n \to z$. There is nothing to prove if $x_n = z$ for some n, so that we may assume $x_n \neq z$. Without loss of generality we may also assume that $x_n \in J_2$ approaches z from one side, i.e., that we can choose the arcs $[z,x_n]$ of J_2 such that

(A.12) $$[z,x_n] \downarrow [z,z] = \{z\} , \; x_n \neq z.$$

Furthermore, there exist $y_n \in J_2 \cap A_2$ and choices of the arcs $[x_n,y_n]$ on J_2 from x_n to y_n such that

(A.13) $$(x_n,y_n) \subset \mathrm{int}(J_1) .$$

Since A_2 and A_4 are separated on J_1 by A_1 and A_3 we must have $A_2 \cap A_4 \subset A_1 \cup A_3$ and

$$J_2 \cap A_2 \cup A_4 \subset J_2 \cap (A_1 \cup A_3) = \emptyset \quad (\text{by (A.11)}).$$

Therefore $y_n \in J_2 \cap A_2$ is bounded away from $z \in J_2 \cap A_4$. In addition, from (A.12) and (A.13) the arc $[x_n, y_n]$ does not contain the point $z \in A_4 \subset J_1$. It follows that from some n_0 on the arcs $[z, x_n]$ and

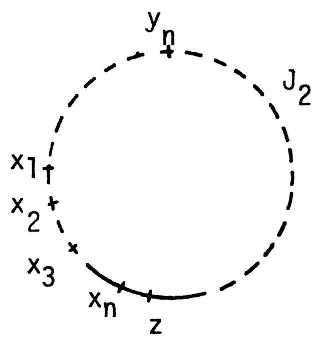

Figure A.5. The location of some points on J_2. y_n cannot lie in the solidly drawn segment.

$[x_n, y_n]$ only have the point x_n in common, and $x_{n_0} \in (x_n, y_n)$. But then, by virtue of (A.12)

$$(z, x_{n_0}] = \bigcup_{n \geq n_0} (x_n, x_{n_0}] \subset \bigcup_{n \geq n_0} (x_n, y_n) \subset \operatorname{int}(J_1) .$$

Consequently also

$$(z, y_{n_0}) = (z, x_{n_0}] \cup (x_{n_0}, y_{n_0}) \subset \operatorname{int}(J_1)$$

so that $z \in G_2$. This proves that G_2 is closed and the same proof works for G_4.

Next we take for A'_r, $r = 2,4$, a closed subarc of A_r which contains the common endpoint of A_r and A_1, but not the common endpoint of A_r and A_3, and which is such that

$$J_2 \cap A_r \subset A'_r . \tag{A.14}$$

Such A'_r exist since $J_2 \cap A_3 = \emptyset$ (by (A.7)). Note that by (A.7) also $J_2 \cap A_1 = \emptyset$ so that A_2 and A_4 must have nonempty interiors. We can and shall therefore also take the interiors of A'_2 and A'_4 nonempty. Now define

$$F_2 = G_2 \cup A'_2 ,$$

$$F_4 = G_4 \cup A'_4 \cup A_1 .$$

Since A_1, A'_2 and A'_4 and G_r are closed, F_2 and F_4 are closed.

First we assume

(A.15) $$G_2 \cap G_4 \neq \emptyset .$$

We can then find an $x_0 \in G_2 \cap G_4 \subset J_2 \cap \bar{J}_1$ and points $y_r \in J_2 \cap A_r$ and arcs $[x_0,y_r]$ of J_2 from x_0 to y_r such that $(x_0,y_r) \subset \text{int}\, J_1$, $r = 2,4$. Note that automatically $y_r \in \mathring{A}_r$ since by (A.7)

$$J_2 \cap A_r \subset \mathring{A}_r \ , \ r = 2,4.$$

If $x_0 \in A_2$, then the arc $[x_0,y_4]$ satisfies all requirements for B and we are done. Similary if $x_0 \in A_4$. $x_0 \in J_2 \cap (A_1 \cup A_3)$ is impossible, by virtue of (A.11). Since $J_1 = A_1 \cup A_2 \cup A_3 \cup A_4$ this takes care of $x_0 \in J_1$, and leaves us with $x_0 \in J_2 \cap \text{int}(J_1)$. In this case, the arc $[x_0,y_r]$ hits J_1 first in $J_2 \cap A_r$ (at y_r), and neither of the arcs $[x_0,y_2]$ and $[x_0,y_4]$ can be a subarc of the other. Thus $[x_0,y_2]$ and $[x_0,y_4]$ only have the point x_0 in common and we can take $B = [x_0,y_2] \cup [x_0,y_4]$. This is the arc of J_2 from y_2 to y_4 through x_0, with

$$\mathring{B} = (x_0,y_2) \cup (x_0,y_4) \cup \{x_0\} \subset \text{int}(J_1).$$

Thus, in this case the lemma is again true, and we have found B whenever (A.15) holds.

Now assume that

(A.16) $$G_2 \cap G_4 = \emptyset .$$

We shall complete the proof by showing that (A.16) leads to a contradiction. Denote by a the common endpoint of A_1 and A_2 (see Fig. A.4). If (A.16) holds, then

(A.17) $$G_2 \cap (A_4' \cup A_1) = \emptyset ,$$

since $G_2 \subset J_2$ implies

$$G_2 \cap A_1 \subset J_2 \cap A_1 = \emptyset \quad \text{(by (A.7))},$$

$$G_2 \cap A_4' \subset G_2 \cap J_2 \cap A_4 \subset G_2 \cap G_4 = \emptyset \quad \text{(by (A.10))}.$$

Similary $G_4 \cap A_2' = \emptyset$ so that

(A.18) $$F_2 \cap F_4 = A_2' \cap (A_4' \cup A_1) = \{a\} .$$

Next we choose a point $b \in \text{int}(J_1) \cap \text{int}(J_2)$ sufficiently close to a, so that we may connect b to $A_2' \setminus \{a\}$ and to $A_1 \cup A_4' \setminus \{a\}$ by continuous paths ϕ_2 and ϕ_4, respectively, which are contained in $int(J_1) \cap int(J_2)$ except for the final point of ϕ_2 which lies on A_2' and the final point of ϕ_4 which lies on $A_1 \cup A_4'$. This can be done because $a \in A_1 \subset int(J_2) \cap J_1$, and by exercise VI.18.3 in Newman (1951) we may assume that A_2' and $A_1 \cup A_4'$ are segments radiating from $a \in J_1 \cap int(J_2)$; note that A_2' and A_4' have nonempty interiors by construction. Finally, let $c \in A_3$. We can then connect b to c by the following curve π_2: Go from b to A_2' along ϕ_2 and then continue along $A_2 \cup A_3 \setminus \{a\}$ to c. This path is disjoint from F_4 because $A_2 \cup A_3 \setminus \{a\}$ and $A_4' \cup A_1$ are disjoint, while ϕ_2 minus its final point lies in $int(J_1) \cap int(J_2)$ which is disjoint from $F_4 \subset J_1 \cup J_2$, and finally

$$(A_2 \cup A_3 \setminus \{a\}) \cap G_4 \subset (A_2 \cap J_2 \cap G_4) \cup (A_3 \cap J_2) \subset G_2 \cap G_4 = \emptyset$$

(compare proof of (A.17)).

In the same way we can connect b with c by a path which moves along ϕ_4, and $A_1 \cup A_4 \cup A_3 \setminus \{a\}$, and which does not intersect F_2. Since $F_2 \cap F_4$ is connected (see (A.18)), Alexander's lemma (Newman (1951), Theorem V.9.2) implies that b is connected to c by a continuous curve ψ disjoint from $F_2 \cup F_4$. This, however, is impossible as we now show. ψ begins at $b \in int(J_1) \cap int(J_2)$ and ends at $c \in A_3 \subset ext(J_2) \cap J_1$. Let d be the first point of ψ in J_1. Then, since ψ is disjoint from $F_2 \cup F_4$, we must have

$$\text{(A.19)} \qquad d \in A_3 \cup (A_2 \setminus A_2') \cup (A_4 \setminus A_4').$$

The right hand side of (A.19) lies in $ext(J_2)$ by (A.7) and the fact that $A_r \setminus A_r'$ is (by (A.14)) disjoint from J_2 and contains the common endpoint of A_r and A_3 in $ext(J_2)$. Therefore, in going from $b \in int(J_1) \cap int(J_2)$ to d along ψ we must hit J_2 in a point $e \in J_2 \cap \bar{J}_1$ (because d is the first point of ψ on J_1). But any such point e must lie in $F_2 \cup F_4$ since we can go from e along some arc of J_2 to $ext(J_1)$ ($J_2 \subset \bar{J}_1$ is impossible by (A.7)). If this arc hits A_2 before A_4 then $e \in G_2$, and if it hits A_4 before A_2 then $e \in G_4$. Thus ψ must intersect $F_2 \cup F_4$ and we have deduced a contradiction from (A.16). □

Proposition 2.2. Let J be a Jordan curve on $\mathcal{M}$ (and hence also on $\mathcal{G}$ and on $\mathcal{G}^*$) which consists of four closed arcs A_1, A_2, A_3, A_4 with disjoint interiors, and such that A_1 and A_3 each contain at least one vertex of $\mathcal{M}$. Assume that one meets these arcs in the order A_1, A_2, A_3, A_4 as one traverses J in one direction. Then there exists a path r on $\mathcal{G}$ inside $\bar{J} = J \cup \text{int}(J_1)$ from a vertex on A_1 to a vertex on A_3, and with all vertices of r in $\bar{J} \setminus A_1 \cup A_3$ occupied, if and only if there does not exist a vacant path r^* on $\mathcal{G}^*$ inside $\bar{J} \setminus A_1 \cup A_3$ from a vertex of $\mathring{A}_2$ to a vertex of $\mathring{A}_4$.

Proof: First assume that there exists a vacant path r^* on $\mathcal{G}^*$ inside $\bar{J} \setminus A_1 \cup A_3$ from $y_2 \varepsilon \mathring{A}_2$ to $y_4 \varepsilon \mathring{A}_4$. Since A_2 and A_4 separate A_1 and A_3 on J any path r inside $\bar{J}$ from a vertex on A_1 to a vertex on A_3 must intersect r^* (e.g. by Newman (1951) Theorem V.11.8). If r is on $\mathcal{G}$ and r^* on $\mathcal{G}^*$, then they must intersect in a vertex of $\mathcal{M}$ (and of $\mathcal{G}$ and $\mathcal{G}^*$) by Comment 2.2(vii). This vertex would lie in $\bar{J} \setminus A_1 \cup A_3$ and be vacant, as a vertex of r^*. Thus any path on $\mathcal{G}$ in $\bar{J}$ connecting a vertex on A_1 with a vertex of A_3 would have to contain a vacant vertex in $\bar{J} \setminus A_1 \cup A_3$. Consequently, no path r as required in the lemma exists. This proves one direction of the proposition.

Now for the converse. Without loss of generality we may assume that the plane has been mapped homeomorphically onto itself such that J is now the unit circle, that A_1 (A_3) intersects the line segment

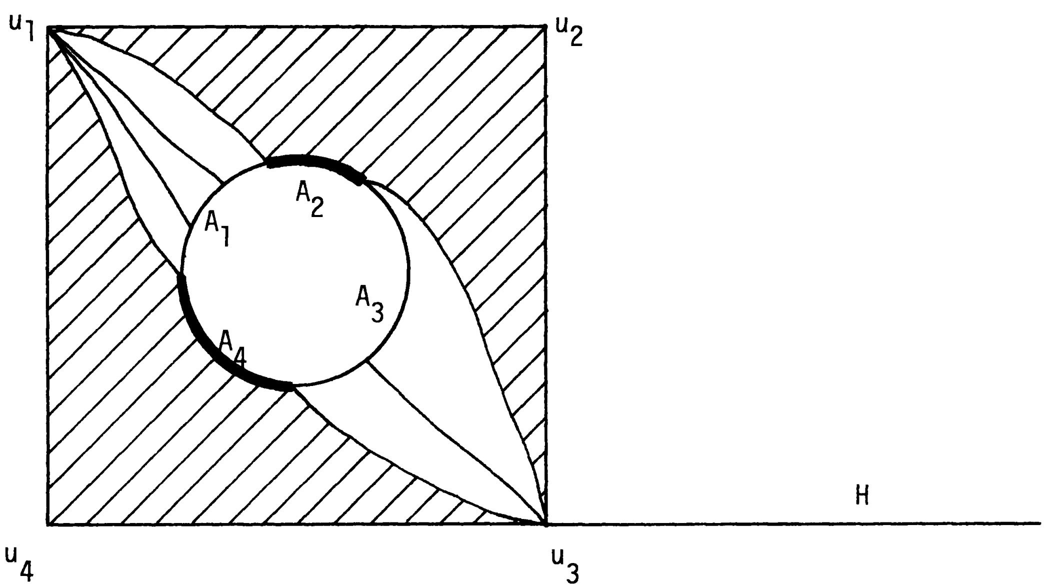

Figure A.6. J is the circle in the center. A_2 and A_4 are the boldly drawn arcs. The two hatched regions are two faces of $\mathcal{M}_1$.

from the origin to $(-2,+2)$ (to $(2,-2)$), while A_2 (A_4) lies between A_1 and A_3 (A_3 and A_1) as we go around J clockwise. We next modify the graphs outside $\bar{J}$, as well as the occupancy configuration outside $\text{int}(J)$. We shall then apply Cor.2.2 to the modified graph and configuration. The mosaic $\mathcal{M}$ is modified to a mosaic $\mathcal{M}_1$ as follows. The vertices of $\mathcal{M}_1$ are the vertices of $\mathcal{M}$ in $\bar{J}$ plus all points of the form $(2i_1,2i_2)$, $i_1,i_2 \in \mathbb{Z}$. As for edges, there is an edge of $\mathcal{M}_1$ between $(2i_1,2i_2)$ and the four points $(2i_1 \pm 2,2i_2 \pm 2)$. The edges of $\mathcal{M}$ in $\bar{J}$ are also edges of $\mathcal{M}_1$. Finally, we write

$$u_1 = (-2,2),\ u_2 = (2,2),\ u_3 = (2,-2),\ u_4 = (-2,-2)$$

and we give $\mathcal{M}_1$ an edge between u_r and any vertex on A_r, $r = 1$ or 3 (see Fig. A.6). $\mathcal{M}_1$ has no further edges. We insert the edges from A_r to u_r in such a way that they lie in $\text{int}(S_1)\setminus\bar{J}$, except for their endpoints, where S_1 is the square

$$S_1 = \{(x_1,x_2): |x_1| \le 2, |x_2| \le 2\}\ .$$

Moreover, we choose these edges such that an edge from A_1 to u_1 and an edge from A_3 to u_3 do not intersect, while the edges from A_r to u_r intersect in u_r only (see Fig. A.6). Thus $\mathcal{M}_1$ contains a copy of the mosaic $\mathcal{M}$ of Ex. 2.2(i) (multiplied by a factor two). In $\bar{J}$ $\mathcal{M}_1$ coincides with the original $\mathcal{M}$, while there are no edges in $S_1\setminus\text{int}(J)$ which have interior intersections. The faces of $\mathcal{M}_1$ are the open squares into which the plane is divided by the lines $x_1 = 2i_1$, $x_2 = 2i_2$, $i_1,i_2 \in \mathbb{Z}$ - with the exclusion of $\mathring{S}_1$ - as well as the faces of $\mathcal{M}$ inside J, plus certain faces in $\mathring{S}_1\setminus\bar{J}$. The last kind of faces are either "triangular" bounded by two edges from u_r to A_r and an edge of $\mathcal{M}$ in A_r, or a face bounded by the two edges on the perimeter of S_1 incident to u_s, $s = 2,4$, one edge from u_1 to A_1 and one from u_3 to A_3 plus an arc of J containing A_s (these are the hatched faces in Fig. A.6). It is clear that $\mathcal{M}_1$ is a mosaic.

We next take for $\mathcal{F}_1$ the collection of faces of $\mathcal{M}$ in $\bar{J}$ which belong to $\mathcal{F}$. In other words, a face F of $\mathcal{M}_1$ belongs to $\mathcal{F}_1$ iff $F \subset \text{int}(J)$ (in which case F is also a face of $\mathcal{M}$) and $F \in \mathcal{F}$. Note that since J is a Jordan curve made up from edges of $\mathcal{M}$, which are also edges of $\mathcal{M}_1$, each face of $\mathcal{M}$ and of $\mathcal{M}_1$ lies either entirely in $\text{int}(J)$ or in $\text{ext}(J)$. We take $(\mathcal{G}_1,\mathcal{G}_1^*)$ as the matching pair based on $(\mathcal{M}_1,\mathcal{F}_1)$. Clearly $\mathcal{G}_1$ and $\mathcal{G}_1^*$ coincide with $\mathcal{G}$ and $\mathcal{G}^*$, respectively, in $\bar{J}$.

Finally we define the modified occupancy configuration on $\mathcal{M}_1$. Let ω be the original occupancy configuration on $\mathcal{M}$. Let H be the half line from u_3 parallel to the first coordinate axis: $H = \{(x_1,x_2): x_1 \geq 2, x_2 = -2\}$. Then we take

$$\text{(A.20)} \qquad \begin{aligned} \omega_1(v) &= \omega(v) \quad \text{if} \quad v \in \bar{J} \setminus A_1 \cup A_3 , \\ \omega_1(v) &= +1 \quad \text{if} \quad v \in A_1 \cup A_3 \cup H , \\ \omega_1(v) &= -1 \quad \text{if} \quad v \notin \bar{J} \text{ and } v \notin H. \end{aligned}$$

We choose a vertex v in A_1 and take

> $W_1 = W_1(v,\omega_1)$ = occupied component of v on $\mathcal{G}_1$ in the configuration ω_1.

Now assume that there does not exist any path r on $\mathcal{G}$ in $\bar{J}$ from a vertex on A_1 to a vertex on A_3 with all vertices on r and in $\bar{J}\setminus A_1 \cup A_3$ occupied. In this case W_1 cannot contain any point on A_3. For if there would be an occupied path r_1 on $\mathcal{G}_1$ from v to a vertex of A_3, then either r_1 is contained in $\bar{J}$ or it leaves $\bar{J}$ before it reaches A_3. The first case cannot arise, for if r_1 stays in $\bar{J}$, then r_1 is also a path on $\mathcal{G}$ and the vertices on r_1 in $\bar{J}\setminus A_1 \cup A_3$ would also have to be occupied in ω ($\omega(v) = \omega_1(v)$ for all such vertices; see (A.20)). Thus, the piece of r_1 from its last vertex on A_1 to its first vertex on A_3 would be a path r of the kind which we just assumed not to exist. Also the second case is impossible, because the only way to leave $\bar{J}$ on $\mathcal{G}_1$ without hitting A_3 is via u_1 and u_1 is vacant in ω_1 by (A.20). Thus no occupied path r_1 can go through u_1. It follows that indeed $W_1 \cap A_3 = \emptyset$. Since all vertices of $A_3 \cup H$ are occupied in ω_1, and can therefore be connected by occupied paths on $\mathcal{G}_1$ in ω_1, it follows that they belong to one component, and

$$\text{(A.21)} \qquad W_1 \cap (A_3 \cup H) = \emptyset.$$

Since all vertices outside $\bar{J}$ and not on H are vacant we obtain also $W_1 \subset \bar{J}$.

We are now ready to apply Cor. 2.2. This Corollary, applied to the cluster W_1 on $\mathcal{G}_1$ shows that there exists a vacant circuit J^* on $\mathcal{G}_1^*$ surrounding W_1. Now all vertices on A_1 are occupied in ω_1 (see (A.20)) and hence belong to W_1 (since $v \in A_1$). Thus

(A.22) $$A_1 \subset W_1 \subset \text{int}(J^*).$$

Also, J^* being vacant cannot intersect $A_3 \cup H$, since it would then have to intersect this set in a vertex (see Comment 2.2(vii)) and all vertices on $A_3 \cup H$ are occupied in ω_1. But since H goes out to ∞ and $A_3 \cup H$ together with the edges from A_3 to u_3 form a connected set, this means that

(A.23) $$A_3 \cup H \subset \text{ext}(J^*).$$

We can now apply Lemma A.2 with $J_1 = J$, $J_2 = J^*$ - (A.22) and (A.23) correspond to (A.7). J^* therefore must contain an arc B such that $\mathring{B} \subset \text{int}(J) \subset \bar{J} \backslash A_1 \cup A_3$ and one endpoint on each of $\mathring{A}_2$ and $\mathring{A}_4$. The arc B therefore lies in $\bar{J} \backslash A_1 \cup A_3$ and in this region $\mathcal{G}_1^*$ coincides with $\mathcal{G}^*$ and ω_1 with ω. Thus all vertices of $\mathcal{G}^*$ on B are vacant. Also, all points of B belong to edges of $\mathcal{G}^*$ in $\bar{J} \backslash A_1 \cup A_3$, because J^* is a circuit on $\mathcal{G}^*$. The endpoints of B belong to $J^* \subset \mathcal{G}^*$, as well as to $J \subset \mathcal{G}$ (since $\mathring{A}_2 \cup \mathring{A}_4 \subset J$), hence are necessarily vertices of $\mathcal{G}^*$ (see Comment 2.2(vii)). It follows that B is made up of the complete edges of a vacant path r^* on $\mathcal{G}^*$ inside $\bar{J} \backslash A_1 \cup A_3$, and runs from a vertex on $\mathring{A}_2$ to a vertex $\mathring{A}_4$. The existence of such an r^* was just what we wanted to prove. □

We remind the reader of the set up for Proposition 2.3. J is a Jordan curve consisting of four nonempty closed arcs B_1, A, B_2, C with A and C separating B_1 and B_2 on J. $L_i : x(1) = a_i$, $i = 1,2, a_1 < a_2$, are two axes of symmetry for $\mathcal{G}_{p\ell}$, and for $i = 1,2$

(A.24) B_i is a curve made up from edges of $\mathcal{M}_{p\ell}$, or B_i lies on L_i and J lies in the halfplane $(-1)^i (x(1) - a_i) \leq 0$.

The proposition deals with paths $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$

(A.25) $v_1, e_2, \dots, e_{\nu-1}, v_{\nu-1} \subset \text{int}(J)$,

(A.26) e_1 has exactly one point in common with J. This lies in B_1 and is either v_0, or in case $B_1 \subset L_1$ it may be the midpoint of e_1,

and

(A.27) e_ν has exactly one point in common with J. This lies in B_2 and is either v_ν, or in case $B_2 \subset L_2$, it may be the midpoint of e_ν.

$J^-(r)$ and $J^+(r)$ are the components of $\text{int}(J)\setminus r$ with A and C in their boundary, respectively. $r_1 \prec r_2$ means $J^-(r_1) \subset J_2^-(r_2)$ (see Def. 2.11 and 2.12). For a path r and a subset S of $\mathbb{R}^2$ $r \subset S$ means that all edges and vertices of r lie in S. We only consider sets S for which

(A.28) $$B_1 \cap B_2 \cap S = \emptyset .$$

Proposition 2.3. Assume that (2.3)-(2.5) hold with $\mathcal{G}$ replaced by $\mathcal{M}$ and that $L_i : x(1) = a_i$, i = 1,2, are axes of symmetry for $\mathcal{G}_{p\ell}$, with $a_1 < a_2$. Let J be a Jordan curve consisting of four closed nonempty arcs B_1 A, B_2 and C as above satisfying (A.24). Let S be any subset of $\mathbb{R}^2$ such that (A.28) holds. Denote by $\mathfrak{R} = \mathfrak{R}(S,\omega)$ the collection of all occupied paths r on $\mathcal{G}_{p\ell}$ which satisfy (A.25) -(A.27) and $r \subset S$. If $\mathfrak{R} \neq \emptyset$, then it has a unique element $R = R(S,\omega)$ which precedes all others. Any occupied path r on $\mathcal{G}_{p\ell}$ which satisfies (A.25)-(A.27) and $r \subset S$ also satisfies

(A.29) $$r \cap \bar{J} \subset \bar{J}^+(R) \quad \text{and} \quad R \cap \bar{J} \subset \bar{J}^-(r).$$

Finally, let r_0 be a fixed path on $\mathcal{G}_{p\ell}$ satisfying (A.25)-(A.27) and $r_0 \subset S$ (no reference to its occupancy is made here). Then, whether $R = r_0$ or not depends only on the occupancies of the vertices of $\mathcal{G}_{p\ell}$ in the set

(A.30) $$(\bar{J}^-(r_0) \cup V_1 \cup V_2) \cap S,$$

where $V_i = \emptyset$ if B_i is made up from edges of $\mathcal{M}_{p\ell}$, while

V_i = {v:v a vertex of $\mathcal{G}_{p\ell}$ such that its reflection $\tilde{v}$ in L_i belongs to $\bar{J}^-(r_0)$ and such that $e \cap \bar{J} \subset \bar{J}^-(r_0) \cap S$ for some edge e of $\mathcal{G}_{p\ell}$ between v and $\tilde{v}$}, i = 1,2,

in case B_i lies in L_i, but is not made up from edges of $\mathcal{M}_{p\ell}$.

Proof: Assume $\mathfrak{R} \neq \emptyset$ and $r_1, r_2 \in \mathfrak{R}$. We shall first construct a path r on $\mathcal{G}_{p\ell}$ satisfying (A.25)-(A.27) as well as

(A.31) each edge of $\mathcal{G}_{p\ell}$ which appears in r also appears in r_1 or in r_2, and

(A.32) $$r \prec r_1 \text{ and } r \prec r_2 .$$

Since the vertices on r are endpoints of the edges appearing in r, each vertex on r also lies on r_1 or r_2. In particular since $r_1, r_2 \subset S$ (A.31) will imply $r \subset S$. Moreover all vertices on r will be occupied since this holds for $r_1, r_2 \in \mathfrak{R}$. Thus r will be an element of $\mathfrak{R}$ which precedes r_1 and r_2. By carrying out this process repeatedly we obtain paths $r \in \mathfrak{R}$ which occur earlier and earlier in the partial order. After a finite number of steps we shall arive at the minimal crossing R.

Now for the details. Let $r_1 = (v_0, e_1, \ldots, e_\nu, v_\nu)$ and $r_2 = (w_0, f_1, \ldots, f_\tau, w_\tau)$. Both of these paths are self-avoiding, so that the curve C_1 made up from $e_1, \ldots, e_\nu$ is a simple arc with endpoints v_0 and v_ν. C_1 intersects J in exactly two points, $m_0 \in B_1$ and $m_\nu \in B_2$. m_0 equals v_0 or the midpoint of e_1, and m_ν equals v_ν or the midpoint of e_ν. The open arc of C_1 between m_0 and m_ν lies in $\text{int}(J)$. Similar comments apply to the curve C_2 made up from the edges of $r_2 : f_1, \ldots, f_\tau$.

If C_2 contains no point in $J^-(r_1)$ then we take $r = r_1$. We shall see below (after (A.44)) that this implies (A.32). ((A.25)-(A.27) and (A.31) are obvious in this case). Let us therefore assume that C_2 contains a point $x \in J^-(r_1)$. Then x belongs to some edge of r_2, say $x \in f_\alpha$. We note that all edges of r_1 and r_2 are edges of the planar graph $\mathcal{G}_{p\ell}$. Two such edges, if they do not coincide, can intersect only in a vertex of $\mathcal{G}_{p\ell}$, which is a common endpoint of these edges. Thus an edge f of r_2 which contains a point of $J^-(r_1)$ cannot leave $J^-(r_1)$ by crossing r_1. If it crosses $\text{Fr}(J^-(r_1)) \setminus r_1$ then it crosses J and f must be f_1 or f_τ, and f intersects J only once, in the midpoint of f. In this case one half of f lies in $\text{ext}(J) \cup J$ while the interior of the other half - which contains a point of $J^-(r_1)$ - must lie entirely in $J^-(r_1)$ (cf. Comment 2.4(ii)). Thus for any edge f of r_2 we must have

(A.33) $$\text{either } \mathring{f} \cap \text{int}(J) \subset J^-(r_1) \text{ or } f \cap \text{int}(J) \subset \bar{J}^+(r_1).$$

In particular

(A.34) $$\mathring{f}_\alpha \cap \text{int}(J) \subset J^-(r_1).$$

Also, if we move along the arc C_2 from x to w_0, then the first intersection with C_1, if any, must be a vertex of $\mathcal{G}_{p\ell}$ which is a common endpoint of an edge of r_2 and an edge of r_1. In particular it must equal v_β for some $0 \le \beta \le \nu$. If such an intersection exists we take b equal to this intersection; if no such intersection exists we take $b = w_0$, the initial point of r_2. Similarly, if moving along C_2 from x to w_τ there is an intersection with C_1 then we take c equal to the first such intersection; otherwise we take $c = w_\tau$, the final point of r_2. In all cases b and c are vertices of r_2, and if c is on r_1, then $c = v_\gamma$ for some $0 \le \gamma \le \nu$. We write ρ for the piece of r_2 between b and c. I.e., if $b = w_\delta$, $c = w_\varepsilon$ with $\delta < \varepsilon$ then $\rho = (w_\delta, e_{\delta+1}, \dots, e_\varepsilon, w_\varepsilon)$, and δ and ε are interchanged when $\delta > \varepsilon$. The same argument used above for showing (A.33) shows that ρ - which contains the point $x \in J^-(r_1)$ - cannot leave $J^-(r_1)$ through r_1, and that if ρ crosses J, then ρ contains a half edge in $\text{ext}(J) \cup J$, the other half being in $J^-(r_1)$. Thus

$$\text{(A.35)} \qquad \mathring{\rho} \cap \text{int}(J) = (\rho \setminus \{b,c\}) \cap \text{int}(J) \subset J^-(r_1).$$

In the sequel we restrict ourselves to the case where $b = w_\delta$ and $c = w_\varepsilon$ with $1 \le \delta < \varepsilon \le \tau-1$. This means that (A.35) simplifies to

$$\text{(A.36)} \qquad \mathring{\rho} = \rho \setminus \{b,c\} \subset J^-(r_1).$$

We leave it to the reader to make the simple changes which are necessary when $b = w_0$ and/or $c = w_\tau$. We define a new path $\tilde{r}_1$ by replacing the piece of r_1 between b and c by ρ. Note that we may have $b = v_\beta = w_\delta$, $c = v_\gamma = w_\varepsilon$ with $\gamma < \beta$. We then have to reverse ρ and in this case $\tilde{r}_1$ becomes

$$\tilde{r}_1 = (v_0, e_1, \dots, e_\gamma, v_\gamma = w_\varepsilon, f_\varepsilon, w_{\varepsilon-1}, \dots, f_{\delta+1}, w_\delta = v_\beta, e_{\beta+1}, \dots, e_\nu, v_\nu).$$

(In the simpler case $\beta < \gamma$ ρ is inserted in its natural order.) We show that $\tilde{r}_1$ is a path satisfying (A.25)-(A.27). $\tilde{r}_1$ consists of one or two pieces of r_1 and ρ. Each of these pieces is a piece of a self-avoiding path, hence self-avoiding. Also, $\mathring{\rho}$ does not intersect r_1, and if $\tilde{r}_1$ contains two pieces of r_1 then they are disjoint (because b and c are distinct, being two points of the simple arc C_2, one strictly before and one strictly after x on C_2). Therefore $\tilde{r}_1$ is self-avoiding. Let $\tilde{r}_1 = (\tilde{v}_0, \tilde{e}_1, \dots, \tilde{e}_\xi, \tilde{v}_\xi)$. Then by construction each of the edges $\tilde{e}_i$, $2 \le i \le \xi-1$, is one of the edges

$e_2,\ldots,e_{\nu-1},f_2,\ldots f_{\tau-1}$, and similarly

$$\{\tilde{v}_1,\ldots,\tilde{v}_{\xi-1}\} \subset \{v_1,\ldots,v_{\nu-1},w_1,\ldots,w_{\tau-1}\}.$$

Thus $\tilde{r}_1$ satisfies (A.25), because r_1 and r_2 do. Also (A.26) and (A.27) hold, because $\tilde{e}_1 = e_1$, $\tilde{e}_\xi = e_\nu$ when $1 \le \delta < \varepsilon \le \tau-1$. (But even when $b = w_0$ (A.26) is easy for then $\tilde{e}_1 = f_1$; similarly for (A.27).)

For brevity denote by $E(r)$ the collection of edges of $\mathcal{G}_{p\ell}$ appearing in r. Then it is clear from the construction that

(A.37) $$E(\tilde{r}_1) \subset E(r_1) \cup E(r_2).$$

(A.37) says that (A.31) holds for $\tilde{r}_1$ instead of r. Since $r_1 \subset \overline{J}^-(r_1)$ by definition, it is also immediate from the construction and (A.35) that

(A.38) $$\tilde{r}_1 \cap \overline{J} \subset \overline{J}^-(r_1).$$

We show that (A.38) implies

(A.39) $$r_1 \cap \overline{J} \subset \overline{J}^+(r_1) \subset \overline{J}^+(\tilde{r}_1)$$

and

(A.40) $$J^-(\tilde{r}_1) \subset J^-(r_1).$$

To see this, observe first that the arc, J_1 say, of J between the points of intersection of r_1 and J, and containing A, is the only part of $\overline{J}^-(r_1)$ on J. By (A.38) the points of intersection of $\tilde{r}_1$ and J must lie on J_1. Consequently the arc of J between these intersection points containing C also contains that arc of J between the intersection points of J and r_1 containing C. The latter arc is just $J\setminus J_1$. Any interior point z_0 of $J\setminus J_1$ lies therefore in $Fr(J^+(r_1)) \cap Fr(J^+(\tilde{r}_1))$. Such interior points exist since the endpoints of $J\setminus J_1$ are the intersections of r_1 with J; these lie on $B_1 \cap S$ and $B_2 \cap S$, respectively, and cannot coincide by virtue of (A.28). Pick a point z_0 in the interior of $J\setminus J_1$. Any point $z_1 \in int(J)$ sufficiently close to z_0 belongs to $J^+(r_1) \cap J^+(\tilde{r}_1)$. Choose such a z_1 and let y be an arbitrary point of $J^+(r_1)$. There then exists a continuous curve ψ from y to z_1 in $J^+(r_1)$. By (A.38) ψ cannot hit $\tilde{r}_1$, and since ψ lies in $J^+(r_1)$ it cannot hit J either. Thus ψ does not hit $Fr(J^+(\tilde{r}_1))$ and ends at $z_1 \in J^+(\tilde{r}_1)$. Thus all of ψ lies in $J^+(\tilde{r}_1)$ and in particular $y \in J^+(\tilde{r}_1)$. Since

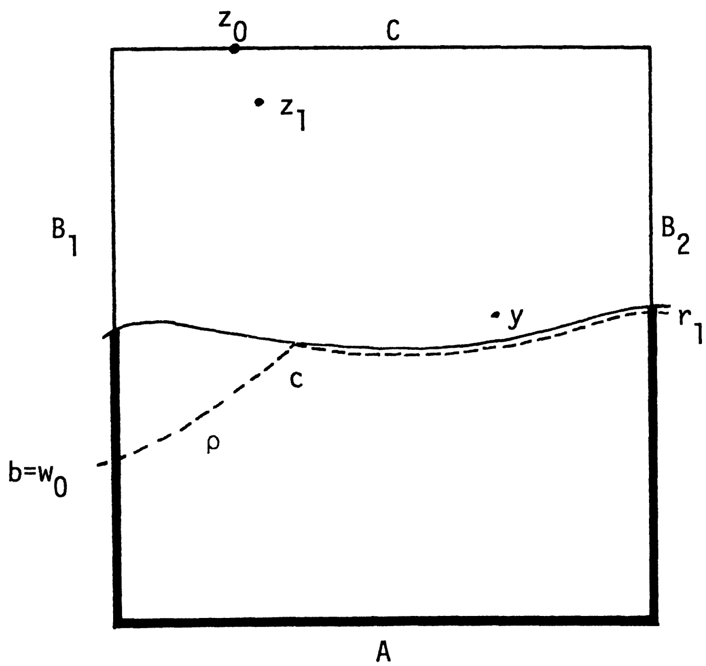

Figure A.7. Schematic diagram giving relative locations J is the perimeter of the rectangle. J_1 is the boldly drawn part of J. r_1 is drawn solidly and $\tilde{r}_1$ is dashed. $\tilde{r}_1$ coincides with $\tilde{r}$ in the part drawn as $\overline{}$ - - - . The figure illustrates a case with $b=w_0$.

y was an arbitrary point of $J^+(r_1)$ we proved

$$J^+(r_1) \subset J^+(\tilde{r}_1). \tag{A.41}$$

The second inclusion in (A.39) follows immediately from this, while the first inclusion in (A.39) is immediate from the definition of J^+. (A.40) follows from (A.39) since $J^-(r) = \text{int}(J)\setminus\overline{J}^+(r)$.

(A.40) implies that if an edge f of r_2 satisfies $\mathring{f} \cap \text{int}(J) \subset J^-(\tilde{r}_1)$, then also $\mathring{f} \cap \text{int}(J) \subset J^-(r_1)$. By virtue of (A.33) the other edges f of r_2 satisfy $f \cap \text{int}(J) \subset \overline{J}^+(r_1)$. f_α is not one of these, by (A.34). However, f_α is part of ρ , and hence of $\tilde{r}_1$ so that $f_\alpha \cap \text{int}(J) \subset \overline{J}^+(\tilde{r}_1)$. Therefore, if we write $N(r)$ for the number of edges f or r_2 with $\mathring{f} \cap \text{int}(J) \subset J^-(r)$, then f_α is counted in $N(r_1)$ but not in $N(\tilde{r}_1)$. Moreover, by the preceding observation, any f counted in $N(\tilde{r}_1)$ must also be counted in $N(r_1)$. Thus

(A.42) $$N(\tilde{r}_1) < N(r_1).$$

We now replace r_1 by $\tilde{r}_1$ and repeat the procedure, if necessary. If C_2 still contains a point in $J^-(\tilde{r}_1)$ then we form $\tilde{r}_2$ such that

$$E(\tilde{r}_2) \subset E(\tilde{r}_1) \cup E(r_2) \subset E(r_1) \cup E(r_2), \quad \text{(cf. (A.37))},$$

$$J^-(\tilde{r}_2) \subset J^-(\tilde{r}_1) \subset J^-(r_1) \quad \text{(cf. (A.40))},$$

and

$$N(\tilde{r}_2) < N(\tilde{r}_1) < N(r_1) \quad \text{(cf. (A.42))}.$$

Since r_2 has finitely many edges $N(r_1) < \infty$, and N decreases with each step. Thus, after a finite number of steps, say λ steps, we arrive at a path $\tilde{r}_\lambda$ satisfying (A.25)-(A.27) and

(A.43) $$E(\tilde{r}_\lambda) \subset E(\tilde{r}_{\lambda-1}) \cup E(r_2)\ldots \subset E(r_1) \cup E(r_2),$$

(A.44) $$J^-(\tilde{r}_\lambda) \subset J^-(\tilde{r}_{\lambda-1}) \subset \ldots \subset J^-(r_1),$$

and such that C_2 contains no more points in $J^-(\tilde{r}_\lambda)$, or equivalently

(A.45) $$r_2 \cap \overline{J} \subset \overline{J}^+(\tilde{r}_\lambda).$$

The case where C_2 contains no points in $J^-(r_1)$ mentioned in the beginning of the proof is subsumed under this, if we take $\tilde{r}_\lambda = r_1$ for this case. We now take $r = \tilde{r}_\lambda$. (A.43) gives us (A.31) while (A.44) and (A.45) give us (A.32). Indeed (A.45) implies $J^-(\tilde{r}_\lambda) = J^-(r) \subset J^-(r_2)$ just as (A.38) implies (A.41) (merely interchange + and -). This completes the construction of r.

Now that we have constructed r from r_1, r_2 the remainder of the proof is easy. Denote the elements of $\mathfrak{R}$ in some order by $r_1, r_2, \ldots, r_\sigma$. If $\mathfrak{R} = \emptyset$ we don't have to prove the existence of R, and when R has only one element, r_1, then $R = r_1$. In general $\mathfrak{R}$ is finite by virtue of (2.3), (2.4). For $\sigma \geq 2$ let r be the path constructed above from r_1 and r_2. For $\sigma = 2$ take $R = r$. For $\sigma \geq 3$ go through the above construction with r_1 and r_2 replaced by r and r_3, respectively. The resulting path, $\overline{r}$ say, is again in $\mathfrak{R}$ and satisfies

$$E(\overline{r}) \subset E(r) \cup E(r_3) \subset E(r_1) \cup E(r_2) \cup E(r_3) \quad \text{(cf. (A.31))}$$

and

$$\overline{r} \prec r_3 \text{ and } \overline{r} \prec r, \text{ hence } \overline{r} \prec r_i,\ 1 \leq i \leq 3 \quad \text{(cf. (A.32))}.$$

After a finite number of such constructions we obtain a path $R \in \mathcal{R}$ which satisfies

$$E(R) \subset \bigcup_{i=1}^{\sigma} E(r_i),$$

$$R \prec r_i, \; 1 \le i \le \sigma . \tag{A.46}$$

This R precedes all elements of $\mathcal{R}$. (A.46) implies

$$R \cap \bar{J} \subset \bar{J}^-(R) \subset \bar{J}^-(r_i), \quad 1 \le i \le \sigma ,$$

and hence $r_i \cap \bar{J} \subset \bar{J}^+(R)$ (just as (A.38) implied (A.39)). Thus (A.29) holds. The uniqueness of R is immediate for if $R' \in \mathcal{R}$ also precedes all elements of $\mathcal{R}$, then $R \prec R'$ and $R' \prec R$. Then (A.29) holds for R as well as R' so that

$$R \cap \bar{J} \subset \bar{J}^-(R'), \quad R \cap \bar{J} \subset \bar{J}^+(R'),$$

whence

$$R \cap \bar{J} \subset \bar{J}^-(R') \cap \bar{J}^+(R') = R' \cap \bar{J}.$$

Interchanging R and R' yields $R \cap \bar{J} = R' \cap \bar{J}$, which together with (A.26) and (A.27) leads to $R = R'$.

Finally, if r_0 is a path on $\mathcal{G}_{p\ell}$ satisfying (A.25)-(A.27) and $r_0 \subset S$, then $R = r_0$ if and only if $r_0 \in \mathcal{R}$ but r_0 is not preceded by any other element of $\mathcal{R}$. Thus $R = r_0$ is equivalent to

(A.47) r_0 is occupied, but any path r on $\mathcal{G}_{p\ell}$ satisfying (A.25)-(A.27) with $r \subset S$ with $r \prec r_0$, $r \neq r_0$ cannot be occupied.

Clearly, (A.47) only depends on the occupancies of sites on r_0 or on paths $r \prec r_0$ with $r \subset S$. But all such sites belong to $\bar{J}^-(r_0) \cap S$ or are an initial or final point in $\text{ext}(J)$ of a path $r \prec r_0$ with $r \subset S$. Since r has to satisfy (A.26) and (A.27) one easily sees that all these sites belong to the set (A.30)(cf. Comment 2.4(ii)). □

We next prove a purely graph-theoretical proposition, which is needed only in Ch. 9. It was first proved by Sykes and Essam (1964). We find it somewhat simpler to prove the version below which refers to $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ rather than $\mathcal{G}$ and $\mathcal{G}^*$. We remind the reader of the definition of $\mathcal{G}_{p\ell}$ (ω; occupied) for an occupancy configuration ω on $\mathcal{M}_{p\ell}$ satisfying (2.15) and (2.16). $\mathcal{G}_{p\ell}$ (ω; occupied) is the graph with vertex set the set of occupied vertices of $\mathcal{G}_{p\ell}$ and edge set the set of edges of $\mathcal{G}_{p\ell}$ both of whose endpoints are occupied. $\mathcal{G}^*_{p\ell}$ (ω;vacant) is defined similarly; see the proof of Theorem 9.2.

Proposition A.1. Let ω be a fixed occupancy configuration on $\mathcal{M}_{p\ell}$, satisfying (2.15) and (2.16). Two vacant vertices of $\mathcal{G}^*_{p\ell}$ v_1 and v_2 lie in the same component of $\mathcal{G}^*_{p\ell}$ (ω; vacant) if and only if v_1 and v_2 lie in the same face of $\mathcal{G}_{p\ell}$ (ω; occupied).

Proof: v_1 and v_2 lie in the same component of $\mathcal{G}^*_{p\ell}$ (ω; vacant) iff there exists a vacant path on $\mathcal{G}^*_{p\ell}$ from v_1 to v_2. If such a path exists, then it cannot intersect any edge of $\mathcal{G}_{p\ell}$ (ω; occupied) (by virtue of Comment 2.3(v)) so that the path lies entirely in ine face of $\mathcal{G}_{p\ell}$ (ω; occupied). Thus in one direction the proposition is trivial.

For the converse, assume $v_1, v_2 \in \mathcal{G}^*_{p\ell}$ are vacant and lie in the same face of $\mathcal{G}_{p\ell}$ (ω;occupied). By definition of such a face as a component of $\mathbb{R}^2 \backslash \mathcal{G}_{p\ell}$ (ω;occupied) this means that there exists a continuous curve ψ in $\mathbb{R}^2 \backslash \mathcal{G}_{p\ell}$ (ω;occupied) from v_1 to v_2. In order to complete the proof we show how one can modify ψ so that it becomes a path on $\mathcal{G}^*_{p\ell}$ (ω; vacant). To make this modification we recall that all faces of $\mathcal{M}_{p\ell}$ are "triangles" (Comment 2.3(vi)). Assume that ψ intersects such a face, say the open triangle F with distinct vertices w_1, w_2, w_3 and edges e_1 between w_2 and w_3, e_2 between w_3 and w_1, and e_3 between w_1 and w_2. Moving from v_1 to v_2 along ψ let x_1 (x_2) be the first (last) intersection with $\bar{F}$. The x_i are necessarily on the perimeter of F, since both endpoints of ψ are vertices of $\mathcal{G}^*_{p\ell}$, hence not in any of the open triangular faces of $\mathcal{M}_{p\ell}$. If $x_i \in e$, then at least one endpoint of e must be vacant, for otherwise e belongs to $\mathcal{G}_{p\ell}$ (ω; occupied), while ψ is disjoint from this graph. This implies that x_1 can be connected to x_2 by a simple arc along the perimeter of F, which still does not intersect $\mathcal{G}_{p\ell}$ (ω; occupied). For example, let $x_1 \in e_1$, $x_2 \in e_2$. If the common endpoint w_3 of e_1 and e_2 is vacant, then move from x_1 to w_3 along e_1 and from w_3 to x_2 along e_2. If w_3 is occupied, then w_1 and w_2 must be vacant, and one can go from x_1 to w_2 along e_1, from w_2 to w_1 along e_3, and from w_1 to x_2 along e_2. These connections from x_1 to x_2 do not intersect $\mathcal{G}_{p\ell}$ (ω; occupied), because if an edge e does not belong to $\mathcal{G}_{p\ell}$ (ω; occupied), then no interior point of e can belong to $\mathcal{G}_{p\ell}$ (ω; occupied). ψ intersects only finitely many faces, say $F_1,\ldots,F_\nu$. We can successively replace the piece of ψ between the first and last intersection of $\overline{F}_i$ with a simple arc along the perimeter of F_i. Making such a replacement cannot introduce a new face whose interior is entered by ψ. On the contrary, each such replacement diminishes the number of such faces. Consequently, after a finite number of steps we obtain a continuous curve, ϕ say, from v_1 to v_2, disjoint from $\mathcal{G}_{p\ell}$ (ω; occupied), and which is contained in the union of the edges of $\mathcal{M}_{p\ell}$. ϕ may not be a path on $\mathcal{G}^*_{p\ell}$. For instance it can contain only part of an edge e, rather than the whole edge e, and ϕ is not necessarily simple. Note, however, that ϕ begins at the vertex v_1 of $\mathcal{G}^*_{p\ell}$, and ends at v_2 which we may take different from v_1 (there is nothing to prove if $v_1 = v_2$). Let w_1 be the first vertex of $\mathcal{M}_{p\ell}$ different from v_1 through which ϕ passes. Set

$$t_0 = \max\{t \in [0,1]: \phi(t) = v_1\},$$

$$t_1 = \min\{t \in [0,1]: \phi(t) = w_1\}.$$

We can then discard the piece of ϕ from $t = 0$ to $t = t_0$; the restriction of ϕ to $[t_0,1]$ is still a path from v_1 to v_2. Also for $t_0 < t < t$, $\phi(t)$ cannot equal any vertex of $\mathcal{M}_{p\ell}$ and therefore is contained in the union of the interiors of the edges of $\mathcal{M}_{p\ell}$. Since the continuous path ϕ cannot go from the interior of one edge to the interior of another edge without passing through a vertex, this means that $\phi(t)$ for $t_0 < t < t_1$ is contained in the interior of a single edge e_1 from v_1 to w_1. Also by connectedness ϕ passes through all points of e_1. We can therefore replace the piece of ϕ from $t = 0$ to $t = t_1$ by the simple arc e_1. After this replacement ϕ still is a continuous path in $\mathbb{R}^2 \setminus \mathcal{G}_{p\ell}$ (ω; occupied). We repeat this process with w_1 in place of v_1. After a finite number of replacements we obtain a path ρ on $\mathcal{M}_{p\ell} \setminus \mathcal{G}_{p\ell}$ (ω; occupied), with possible double points, from v_1 to v_2. Since ρ does not intersect $\mathcal{G}_{p\ell}$ (ω; occupied) it contains only vacant vertices, and in particular no central vertices of $\mathcal{G}_{p\ell}$ (see (2.15)). Thus ρ is a path with possible double points on $\mathcal{G}^*_{p\ell}$ (ω; vacant). Loop-removal (see Sect. 2.1) from ρ finally yields the required self-avoiding path on $\mathcal{G}^*_{p\ell}$ (ω; vacant) from v_1 to v_2. $\square$

Finally we prove a simple lemma which is used repeatedly, and which guarantees the existence of "periodic paths" resembling straight lines on periodic graphs.

Lemma A.3. Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^d$. Then for each $1 \leq i \leq d$ there exists a vertex $v_0 = (v_0(1),\ldots,v_0(d))$ of $\mathcal{G}$ and a path $r_0 = (v_0,e_1,v_1,\ldots,e_\sigma,v_\sigma)$ on $\mathcal{G}$ such that

(A.48) $$0 \leq v(j) < 1,\ 1 \leq j \leq d,$$

(A.49) $$v_\sigma = v_0+\alpha\xi_i \quad \text{for some integer } \alpha \geq 1$$

and

(A.50) for all $n \geq 1$ the path on $\mathcal{G}$ obtained by successively traversing $r_0+k\alpha\xi_i$, $k = 0,1,\ldots,n$ is a self-avoiding path on $\mathcal{G}$ connecting v_0 with $v_0+(n+1)\alpha\xi_i$.

Proof: Let w_0 be any vertex of $\mathcal{G}$ and r a path on $\mathcal{G}$ connecting w_0 with $w_0+\xi_i$. Then the path on $\mathcal{G}$ obtained by successively traversing $r+k\xi_i$, $k = 0,\ldots,n$ connects w_0 with $w_0+(n+1)\xi_i$, but it may have double points. To get rid of the double points we choose w_1, w_2 on r as follows. First let α be the maximal integer for which there exist vertices w_1, w_2 on r with

(A.51) $$w_2 = w_1+\alpha\xi_i\ .$$

Since the endpoint of r, $w_0+\xi_i$, differs from the initial point of r by ξ_i we see that $\alpha \geq 1$. We now select a pair w_1, w_2 satisfying (A.51) and lying "as close together as possible", in the sense that there does not exist any pair of vertices $(w_3,w_4) \neq (w_1,w_2)$ on the segment of r from w_1 to w_2 with $w_4 = w_3+\alpha\xi_i$. Denote the segment of r from w_1 to w_2 by s. Let $\ell_1,\ldots,\ell_d$ be the unique integers for which $w_0+\sum_1^d \ell_j\xi_j$ lies in the unit cube $[0,1)^d$. We claim that we can take $v_0 = w_0 + \sum_1^d \ell_j\xi_j$ and $r_0 = s + \sum_1^d \ell_j\xi_j$. Since r is self-avoiding so is s and by virtue of periodicity we only have to show that for any $k > 1$ s and $s + k\alpha\xi_i$ cannot intersect, and that the only common point of s and $s + \xi_i$ is $w_2 = w_1 + \alpha\xi_i$, the endpoint of s and initial point of $s + \xi_i$. To see that this is indeed the case consider a vertex w_4 of $\mathcal{G}$ which lies on s as well as on $s + k\alpha\xi_i$. Then $w_3 := w_4 - k\alpha\xi_i$ also lies on s. By our definition of α, this is possible only if $k = 1$. Moreover, if $k = 1$, by our choice of (w_1,w_2) this is possible only if $w_3 = w_1$ and $w_4 = w_2$, as claimed. □