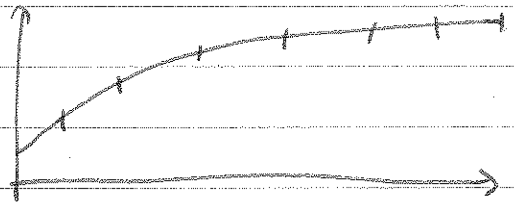
Multifactor IR models

Y^i_t is yield-to-maturity T_i on day t.
These things are often treated 'as is' from a sample MV Gaussian



Warning: Calculate sample cov. matrix V, diagonalise it

$$V \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Suppose $\lambda_1 > \lambda_2 > \dots > \lambda_n$, $\sum \lambda_i = 1$

We may find $\lambda_1 = 0.95$, $\lambda_2 = 0.02$, $\lambda_3 = O(10^{-3})$...
So we just need top 2 eigenvalues..?
Taking top 2 components gives 97% of variance.
But the SD of what remains
$\sqrt{1-0.97} = \approx \underline{0.17}$! not nec. negligable.

We need to consider multifactor models to have any hope of capturing yield curve dynamics...
Let us look at a simple MV Gaussian stories.
We propose that

$$r_t = a + b \cdot X_t$$

where X_t is MV O.U process

$$dX_t = \sigma dB_t - A X_t \, dt$$

A $n \times n$ non-singular and WLOG (by choice of a, b) A diagonal (Maybe Jordan form)

To solve this we proceed as in 1d:

$$d\left(\exp(tA)X_t\right) = e^{tA} dX_t + e^{tA} A X_t dt$$

$$= e^{tA}\sigma dB_t$$

$$\therefore X_t = e^{-tA}X_0 + \int_0^t e^{(s-t)A}\sigma dB_s$$

and thus

$$\int_0^t X_s ds = (I-e^{-tA})A^{-1}X_0 + \int_0^t \left(\int_0^s e^{(u-s)A}\sigma dB_u\right) ds$$

$$= (I-e^{-tA})A^{-1}X_0 + \int_0^t (I-e^{-(t-u)A})A^{-1}\sigma dB_u$$

So we have that

$$\int_0^t X_s ds \sim N\left((I-e^{-tA})A^{-1}X_0, \underbrace{V_t}_{\text{known from above}}\right)$$

and

$$\int_0^t r_s ds = N\left(at + b\cdot(I-e^{-tA})A^{-1}X_0, b\cdot V_t b\right)$$

So in this model the ZCB prices are

$$B(0,t) = \exp\left\{-\left[at + b\cdot(I-e^{-tA})A^{-1}X_0\right] + \tfrac{1}{2} b\cdot V_t b\right\}$$

so the yield to maturity t looks like

$$y(t) = \frac{a + b\cdot(I-e^{-tA})A^{-1}X_0}{t} - \frac{\frac{1}{2} b\cdot V_t b}{t}$$

AFM

How does this look for $N=2$?

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

we get $\exp tA = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ so we see a yield curve that looks like

$$\underbrace{y(t) = a + \frac{1-e^{-\lambda t}}{\lambda t} c_0 + e^{-\lambda t} c_1}_{} + \frac{6 \cdot V_t}{2t}$$

Nelson-Siegel yield curve "model"
- more flexible, but it is only a fit, not a model.

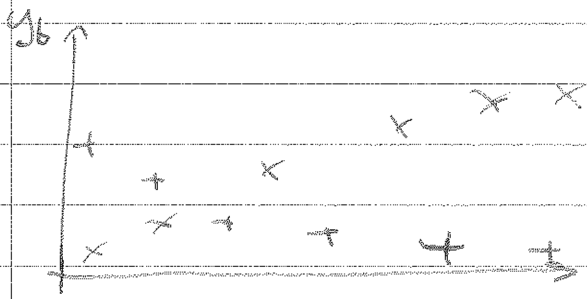
Why do they take A to have Jordan form?

If we had distinct eigenvalues, need to fit

$$y_t = a + \frac{1-e^{-\lambda_1 t}}{\lambda_1 t} c_1 + \frac{1-e^{-\lambda_2 t}}{\lambda_2 t} c_2$$

so $t y_t$ is a lin. comb. of $e^{-\lambda_i t}$, const.

Now look at data.



x x x full data
+ + + residuals after taking out at

If $\lambda_1 > \lambda_2$, can estimate λ_2 from long-term behaviour, but λ_1 could only be estimated at $t \approx 0 \Rightarrow$ not enough data.

Heath-Jarrow-Morton Modelling approach.

The idea here is that we write the bond price in terms of the instantaneous forward rates:

$$B(t,T) = \exp \int_t^T f_{ts}\, ds$$

and then specify dynamics for process $\{f_{ts} : 0 \le t \le s\}$

Note that

$$\lim_{h \downarrow 0} \frac{1 - B(t,t+h)}{h} = \lim_{h\downarrow 0} E_t\left[\frac{1 - e^{-\int_t^{t+h} r_s\, ds}}{h}\right] = r_t$$

under mild conditions.

So $r_t = f_{tt}$

There's a myth that HJM and spot rate models are somewhat different, but they are not.

Now let us suppose we want to build a model in a world with just one Brownian motion. Let us suppose that

$$d f_{ts} = \sigma_{ts}\, dB + \alpha_{ts}$$

so we see that

$$f_{ts} = f_{0s} + \int_0^t \sigma_{us}\, dB_u + \int_0^t \alpha_{us}\, du$$

and so $\int_t^T f_{ts} = \int_t^T f_{0s} + \int_0^t \left(\int_t^T \sigma_{us}\, ds \right) dB_u$

$$+ \int_0^t \left(\int_t^T \alpha_{us}\, ds \right) du$$

Now notice that for T fixed

$$M(t,T) = \mathbb{E}_t\left[\exp\left(-\int_0^T r_s\, ds \right) \right] = \exp\left(-\int_0^t r_s\, ds \right) B(t,T)$$

$$= \exp\left(-\int_0^t \Sigma_{sT}\, dB_s - \tfrac{1}{2} \int_0^t \Sigma_{sT}^2\, ds \right) \underbrace{\exp\left(-\int_0^T f_{0s}\, ds \right)}_{B(0,T)}$$