1 Probability questions

1. Two players roll two standard fair six-sided dice. Player A bets that the sum 12 will appear first, while player B bets that two consecutive sums of 7 will appear first. They keep rolling the dice until one of the two players wins. What is the probability that player A will win?

2. You play the following game. You roll a standard fair six-sided dice, and you have two options: you can either roll it again or you are paid the value you rolled. However, you can roll the dice at most 3 times (so if by the third roll you did not choose to be paid, you are paid the value of the third roll). What strategy should you have in order to earn the most? What is your expected payout in that case? Now assume that you can roll the dice n times. What is your optimal strategy?

3. Fair coin is tossed until three consecutive heads (H) appear. What is the expected number of tosses? What is the expected number of tosses if we toss until the sequence H, T, H (in three consecutive positions) appears? What about the expected number of tosses until (consecutive) H, H, T? Can you generalise to arbitrary sequences of arbitrary length?

4. Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order A, B, C, A,... choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

   Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

6. (a) Show that there exists a constant $c \in (0,1)$ such that the following holds. For any positive integer $n \in \mathbb{N}$, if $G = (V, E)$ is the two-dimensional box of side length n (i.e. graph with vertices $(i,j) \in \{0, 1, \ldots n\}^2$ and edges $\{(i,j),(k,\ell)\}$ iff $|k-i| + |\ell - j| = 1$) then all $A \subset V$ with $|A| = k < \frac{|V|}{2}$ satisfy that the number of edges between A and A^c is at least $c\sqrt{k}$.

   (b) Let $p \in (0,1)$ be a fixed constant. Show that there exists a constant $c \in (0,1)$ such that the following holds: For $n \in \mathbb{N}$ let $G_n = (V_n, E_n)$ be a random graph obtained by taking two copies of two-dimensional boxes of side length n and for each $(i,j) \in \{0, \ldots n\}^2$ with probability $\frac{p}{n}$ adding an edge between the vertex with coordinates (i,j) from the first box to the vertex with the same coordinates (i,j) in the second box, independently of other added edges. Then as $n \to \infty$ probability that G_n is such that all $A \subset V$ with $|A| = k < \frac{|V|}{2}$ satisfy that the number of edges between A and A^c is at least $c\sqrt{k}$ converges to 1.

7. Choose two random numbers from $[0, 1]$ and let them be the endpoints of a random interval. Repeat this n times. What is the probability that there is an interval which intersects all others?

2 Probabilistic method

Some problems in this section can be solved without the use of probability; however, having an approach which is probabilistic gives a quicker solution.

8. Let v_1, v_2,... $v_n \in \mathbb{R}^n$ such that all $|v_n| = 1$. Prove that there exist ε_1,...,$\varepsilon_n = \pm 1$ such that $|\varepsilon_1 v_1 + ... + \varepsilon_n v_n| \leq \sqrt{n}$ and also there exist ε_1,...,$\varepsilon_n = \pm 1$ such that $|\varepsilon_1 v_1 + ... + \varepsilon_n v_n| \geq \sqrt{n}$.

9. Let X be some finite set, and let $E \subset \mathcal{P}(X)$ be some set of subsets of X, such that each of the sets in E is of size exactly n. Prove that if the size of E is not greater than 2^{n-1} then elements of X can be coloured in two colours in such a way that no set in E has all elements of the same colour.

10. Let S be the set of sequences of length 2018 whose terms are in $\{1, 2, 3, 4, 5, 6, 10\}$ and sum up to 3860. Prove that the cardinality of S is at most $2^{3860}(\frac{2018}{2048})^{2018}$.

11. Let $\mathcal{F}$ be a collection of some subsets of $\{1,\ 2,\ 3, ...n\}$ of size $k \leq \frac{n}{2}$, such that each two subsets in $\mathcal{F}$ have a non-empty intersection. Prove that size of $\mathcal{F}$ is at most $\binom{n-1}{k-1}$.

12. We say set A is sum-free if there are no $a_1, a_2, a_3 \in A$ such that $a_1 + a_2 = a_3$. Prove that if B is a set of n non-zero integers, then B contains a sum-free subset A of size greater than $\frac{1}{3}n$.

13. Let $\mathcal{A} \subset \mathcal{P}(X)$ for some set X. We say that $\mathcal{A}$ has property B if elements of X can be coloured in two colours such that no set $A \in \mathcal{A}$ is monochromatic (i.e. no set has only elements of one colour). Let $m(k)$ be smallest number such that there exists set X for which there is $\mathcal{A}$ of size $m(k)$ (i.e. $\mathcal{A}$ is a set of $m(k)$ different subsets of X) which does NOT have property B and for which all sets $A \in \mathcal{A}$ have exactly k elements. Show first that $m(n) \geq \frac{\sqrt{(2n-1)!}}{(n-1)!}$. Show also that $m(n) \geq O(1)2^{n-1}\sqrt{\frac{n}{\ln(n)}}$.

3 Random Walks

14. Consider a simple random walk on a cycle $\mathbb{Z}_n = \{0, 1, \ldots n-1\}$ (where consecutive integers mod n are connected) started from 0. Find the distribution of the last vertex visited by this walk (i.e. for each $k > 0$ find a probability that all other vertices are visited before the vertex k).

15. Let G be a graph, consisting of a finite collection of vertices $V(G)$, some pairs of which are connected by an edge. We declare the neighbours of a vertex v to be those vertices w such that v, w are directly connected by an edge. We assume the graph is connected so that there is a path of edges joining any pair of vertices. Let A be a non-empty subset of the vertices, and $B = V(G) \setminus A$. For each vertex $v \in A$, we declare a value $a_v \in \mathbb{R}$. Use a probabilistic argument to show that there exists a function $f : V(G) \to \mathbb{R}$ such that $f(v) = a_v$ for all $v \in A$, and for all $v \in B$, $f(v)$ is the average of the values taken by f on the neighbours of v. Can you justify that f is unique?

16. Let X_n be a simple random walk on an infinite $d \geq 3$ regular tree. What is the probability that X never returns to its starting vertex? Now consider an infinite tree where each vertex has a degree at least $d \geq 3$ and at most D. Can you bound the probability that the walk ever returns to the starting vertex?

17. Let G be a connected graph on n vertices, $V(G)$, show that

$$t_{\text{hit}} \leq t_{\text{cov}} \leq t_{\text{hit}} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\right)$$

where $t_{\text{hit}} = \max_{x,y \in S} \mathbb{E}_x[T_y]$ where T_y is the hitting time of y and $t_{\text{cov}} = \max_{x \in S} \mathbb{E}_x[T_{\text{cov}}]$ where the cover time T_{cov} is the first time the walk has visited all states [i.e. if the walk is $(X_s)_{s \geq 0}$ then $T_{\text{cov}} = \inf\{t \geq 0 : \forall x \in V(G)\, \exists s \leq t \text{ such that } X_s = x\}$].

4 Continuous random variables

18. Let $(q_k : K \geq 1)$ be a sequence of positive numbers such that $\sum_{k \geq 1} q_k < \infty$. Let $(E_k : k \geq 1)$ be a sequence of independent exponential random variables where the rate (i.e., parameter) of E_k is q_k. Let $E = \inf\{E_k\}$ and let $K = \arg\min\{E_k\}$, with $K = \infty$ if the inf is not attained. Compute $\mathbb{P}(K = k, E > t)$ for all $t > 0$ and $k \geq 1$, and hence identify the joint distribution of K and E. Is the result surprising?

19. (i) Let $E_1, E_2 \ldots$ be i.i.d. $\text{Exp}(\lambda)$. For any fixed $t \in \mathbb{R}^+$ and $k \in \mathbb{N}$, what is the probability that $E_1 + \ldots + E_k \leq t < E_1 + \ldots + E_{k+1}$.
(ii) Number i buses arrive at the bus stop at random times such that the k-th bus arrives at the time $E_1 + \ldots + E_k$ where $E_1, E_2 \ldots$ be i.i.d. $\text{Exp}(i)$. Arrivals of buses with different labels are independent.
(a) What is the probability that exactly 5 buses pass by in 1 hour?
(b) What is the probability that exactly 3 Number 7 buses pass by while I am waiting for a Number 1?
(c) When the maintenance depot goes on strike, half the buses break down before they reach my stop. What then is the probability that I wait for 30 minutes without seeing a single bus?