Numbers and Sets

Most of the questions from this sheet were taken from Imre Leader's introductory sheets.

1 Part 1

1. We have 100 dons, and each is wearing a hat. On each hat is written a number from 1 to N and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the number on his own hat. What is the greatest N for which the dons can guarantee that, whatever the distribution of the numbers, at least one of them guesses correctly?

2. Some red sweets and blue sweets are distributed among 99 bags. Gareth wants to select 50 of the bags in such a way that he obtains at least half of the red sweets and at least half of the blue sweets. Is he always able to do this?

3. For which positive integers k is it possible to place a finite number (but more than zero) of queens on an infinite chessboard in such a way that each queen is attacking exactly k others? What happens if we are allowed an infinite number of queens? (A queen attacks another queen if they are in the same row or column or diagonal, with no other queens in between).

4. A pack of cards is shuffled and dealt out to you one card at a time. At any moment, based on what you have seen so far, you can say "I predict that the next card will be red". (You can only make this prediction once.) Which strategy gives you the greatest chance of being right?

5. Prove that it is impossible to "cube the cube" - in other words, that it is impossible to dissect a cube into a finite number (greater than 1) of cubes, no two of which have the same size.

6. Alice and Bob have a chocolate which is of rectangular shape. They play the following game, taking turns with Alice starting the first. At each step player whose turn it is takes a unit square of the remaining part of the chocolate and takes everything that is right of it and everything that is below it. If they disconnect a chocolate with this move, then they keep the part which contains the top left corner and continue playing with it. The one left with only the top left square loses. Does either of the two players have a winning strategy?

7. Between any two cities, there is an air fare. You start in your home city, and you wish to visit each other city once (without returning home at the end). The "sensible" strategy is: always choose the cheapest flight available (to a city you have not yet visited). The "stupid" strategy is: always choose the most expensive flight available. Can the stupid strategy end up costing less than the sensible strategy?

8. We have an infinite sequence of dons, and each is wearing a hat. The hats are red or blue, and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the colour of his own hat. Can this be done is such a way that, whatever the distribution of hat colours, only finitely many dons guess incorrectly?

9. Let f be a function from reals to $\{0, 1\}$. You are allowed to choose any real number or subset of real numbers and ask what the value of f is in that number, or in each element of the set you chose. You are allowed to ask as many questions as you want and in any order. After you have asked all of your questions, you guess the value of $f(x)$ for some real number x of your choice, which you have not asked about before. Can you do this in such a way that you have the probability 1 of guessing $f(x)$?

10. Let $n \in \mathbb{N}$. We have a deck of $2n$ cards with numbers from 1 to $2n$ written on them. Two magicians are performing the following trick. They let the audience shuffle the cards and place them on the table in some order from left to right, face up. Then the first magician comes to the room and looks at the cards. He is allowed to change the positions of at most two cards by swapping them or to not do anything. Then he turns the cards face down, keeping their relative order, and leaves the room. Another magician comes to the room and the audience chooses a positive integer $k \leq 2n$ and asks him to find a card with the number k written on it, by turning face up at most n cards. Prove that the magicians can arrange a strategy in advance such that the trick is always successful.

11. Each of n elderly dons knows a piece of information not known to any of the others. They communicate by telephone, and in each call the two dons concerned reveal to each other all the information they know so far. What is the smallest number of calls that can be made in such a way that, at the end, all the dons know all the information?

12. If A and B are sets of positive integers, we say that A and B are similar if there is a bijection f from A to B that preserves multiplication, in the sense that $f(xy) = f(x)f(y)$ for all x and y in A. (For example, the set of all positive integers and the set of square numbers are similar, via $f(x) = x^2$.) Among the three sets $A = \{1, 3, 5, 7, 9, 11, ...\}$, $B = \{1, 4, 7, 10, 13, 16, ...\}$ and $C = \{1, 5, 9, 13, 17, 21, ...\}$, are there two that are similar?

13. A cycle in the grid $\mathbb{Z}^d$ means the obvious thing: it consists of a sequence of line segments going from point to neighbouring point (where two points are neighbouring if they differ in exactly one coordinate, and by exactly one in that coordinate), with all points distinct except that the first is the same as the last (and there are more than two points in total). Does there exist a cycle in $\mathbb{Z}^3$ such that none of its projections in the x, y or z directions contains a cycle?

14. Let x and y be positive integers such that for every prime p we have $x \mod p \leq y \mod p$ (where $x \mod p$ denotes the number from $0, 1, 2, ..., p-1$ that x is congruent to mod p). Does it follow that $x = y$?

2 Part 2

15. Let f be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Does there always exist an infinite set $S \subset \mathbb{N}$ such that $f(S^2)$ is *not* the whole of $\mathbb{N}$?

16. In a tournament on n players, each pair play a game, with one or the other player winning (there are no draws). Construct a tournament in which, for any two players, there is a player who beats both of them. Is it true that for any k there is a tournament in which, for any k players, there is a player who beats all of them?

17. Let f be a real function of two real variables. Suppose that for each fixed y the function $f(x, y)$ is a polynomial in x and for each fixed x the function $f(x, y)$ is a polynomial in y. Must $f(x, y)$ be a polynomial in x and y?

18. Let f and g be real polynomials such that the set of values taken by f on the rationals is the same as the set of values taken by g on the rationals. Prove that there exist rationals a and b such that $g(x) = f(ax + b)$.

19. Construct a function f from the rationals to the rationals that takes every value on every interval – in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$. Is there such a function from the reals to the reals?

20. Show that it is impossible to write the open interval $(0, 1)$ as the disjoint union of a family of non-trivial closed intervals.

21. Can the open square $(0, 1)^2$ be written as the disjoint union of a family of non-trivial closed straight-line segments?

22. Let $[a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals whose union contains $[0, 1]$. Does it follow that $\sum_{n=1}^{\infty}(b_n - a_n) \geq 1$?

3 Part 3

23. If $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, must $\sum_{n=1}^{\infty} \frac{x_n}{n}$ be convergent?

24. Let $(x_n)_{n=1}^{\infty}$ be a real sequence with $x_n \to 0$. Prove carefully that we may choose $(\varepsilon_n)_{n=1}^{\infty}$, with each $\varepsilon_n = \pm 1$, such that $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. Does this remain true if the x_n are complex?

25. Let $x_1, x_2, \ldots$ be reals such that $\sum_{n=1}^{\infty} |x_n|$ is convergent. Show that if for every positive integer k we have $\sum_{n=1}^{\infty} x_{kn} = 0$ then $x_n = 0$ for all n. What happens if we drop the restriction that $\sum_{n=1}^{\infty} |x_n|$ is convergent?

26. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for each odd positive integer k, the series $\sum_{n=1}^{\infty} x_n^k$ converges when k belongs to S and diverges when k does not belong to S.

27. A subset of $\mathbb{R}$ is called perfect if it is closed and has no isolated points. Write down a (non-empty) perfect set that does not contain any (non-trivial) interval. Prove that every closed set is the union of a perfect set and a countable set.

28. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable?

29. Let S be a collection of subsets of $\mathbb{N}$ such that for every distinct $A, B \in S$ we have that $A \cap B$ is finite. Can S be uncountable?

30. Let X be a subset of $\mathbb{R}$ such that the only order-preserving injection from X to X is the identity. Must X be finite?