Example Sheet 4

Questions marked with * are optional. Please hand in Questions 3,4 and 14. All Markov chains mentioned in this example sheet are assumed to be irreducible and on a finite state space.

1. * Show that every tree, with a distribution π on it, has at least one and at most two central vertices (completing the proof of Theorem 9.19 about cutoff for Markov chains on trees).

2. Show that for a reversible chain and for all $\varepsilon \in (0, 1/4]$ the following inequalities hold

$$\mathrm{hit}_{1/2}\left(\frac{3\varepsilon}{2}\right) - \lceil 2t_{\mathrm{rel}}|\log(\varepsilon)|\rceil \leq t_{\mathrm{mix}}(\varepsilon) \leq \mathrm{hit}_{1/2}\left(\frac{\varepsilon}{2}\right) + \lceil t_{\mathrm{rel}}|\log(\varepsilon/4)|\rceil$$
$$\mathrm{hit}_{1/2}\left(1-\frac{\varepsilon}{2}\right) - \lceil 2t_{\mathrm{rel}}|\log(\varepsilon)|\rceil \leq t_{\mathrm{mix}}(1-\varepsilon) \leq \mathrm{hit}_{1/2}\left(1-2\varepsilon\right) + \lceil t_{\mathrm{rel}}\log(8)/2\rceil \mathbb{1}_{\{\varepsilon>1/18\}}$$

   and deduce that cutoff and hit-cutoff are equivalent if the chain satisfies $t_{\mathrm{rel}} \ll t_{\mathrm{mix}}$.

3. Consider a reversible transition matrix Q which satisfies for all $x \in S$ that $Q(x,x) = 0$. Let P be the lazy version of Q with some laziness parameter $\alpha \in (0,1)$. Let $c, C \in (0,1)$ be two constants and let $(c_x)_{x\in S}$ be a sequence of values in $[c, C]$. Let $\widetilde{P}$ be a transition matrix such that $\widetilde{P}(x,x) = c_x$ and $\widetilde{P}(x,y) = (1-c_x)Q(x,y)$ for $x \neq y \in S$. Show that the mixing times of $\widetilde{P}$ and P are of the same order.

4. Show that if an aperiodic, reversible Markov chain exhibits cutoff, then the lazy version of this chain also has cutoff. Show also that if a lazy version of a reversible Markov chain exhibits cutoff at time $t_{\mathrm{mix}}^{\mathrm{lazy}}$ and the smallest eigenvalue of the non-lazy version of the nth chain, which we denote by λ_n, satisfies $\frac{1}{1+\lambda_n} \ll t_{\mathrm{mix}}^{\mathrm{lazy}}$ then cutoff also happens for a non-lazy version.

5. Show that for a lazy, reversible chain P we have $t_{\mathrm{mix}} \asymp \max_{x\in S, A\subset S:\pi(A)\geq\frac{1}{2}} \mathbb{E}_x[T_A]$.

6. (a) Let G_n be obtained by considering two copies of complete graphs on n vertices, K_n and connecting them with one edge. Find the order of the mixing time of the lazy simple random walk on G_n.

   (b) Let $\widehat{G}_n$ be a graph obtained by first considering a binary tree (where every vertex has degree three except the leaves and the root, which has degree two) of depth n which has 2^n leaves, connecting all leaves to each other and creating a complete graph K_{2^n} on them, and then taking another copy of K_{2^n} and connecting it with one edge to one of the leaves of the initial binary tree (see figure 1 for illustration when $n = 3$). Show that for all $\alpha > \frac{1}{2}$ the sequence of lazy simple random walks on $\widehat{G}_n$ exhibits hit_α cutoff. Does this sequence exhibit cutoff?

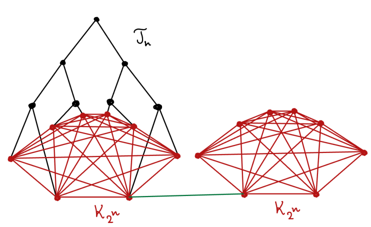

Figure 1: Graph from Question 6 b) for $n = 3$.

7. Is it always true that for a simple random walk on a graph $\mathbb{E}_b[T_a] = \mathbb{E}_a[T_b]$ for all vertices $a, b \in V$?

8. * Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave, i.e. that

$$\frac{1}{2} \cdot (R(r_1) + R(r_2)) \leq R\left(\frac{1}{2}(r_1 + r_2)\right).$$

9. * Without using the commute time identity, show that the effective resistance forms a metric on any network with conductances $(c(e))$.

10. Show that if in a network with source a and sink z, vertices with different voltages are glued together, then the effective resistance from a to z will strictly decrease.

11. Consider a simple random walk on the binary tree of depth k with $n = 2^{k+1} - 1$ vertices (the root has degree two and all other nodes except for the leaves have degree 3).
   (a) Let a and b be two vertices at level m whose most recent common ancestor c is at level $h < m$. Find $\mathbb{E}_a[T_b]$.
   (b) Show that the maximal value of $\mathbb{E}_a[T_b]$ is achieved when a and b are leaves whose most recent common ancestor is the root of the tree.

12. If T_y denotes the first hitting time of y, show that

$$\mathbb{E}_x[T_y] = \frac{1}{2} \sum_z \deg(z) \left(R_{\text{eff}}(x, y) + R_{\text{eff}}(y, z) - R_{\text{eff}}(x, z)\right).$$

13. Show that the effective resistance between any two vertices of an electrical network on a simple graph (i.e. a graph with no multi-edges) with conductances on all edges equal to 1 is upper bounded by the diameter of the graph. Use this to show the following results.
   (a) Prove that for a simple random walk on a simple graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ for all $a, b \in V$ we have

$$\mathbb{E}_a[T_b] + \mathbb{E}_b[T_a] \leq 2nm$$

   and conclude that a lazy simple random walk on G satisfies $t_{\text{mix}} \leq 8nm + 1$.
   (b) Prove that for a simple random walk on a simple d-regular graph $G = (V, E)$ (i.e. $\deg(x) = d$ for all $x \in V$) with $|V| = n$ for all $a, b \in V$ we have

$$\mathbb{E}_a[T_b] + \mathbb{E}_b[T_a] \leq 3n^2 - nd$$

   and conclude that a lazy simple random walk on G satisfies $t_{\text{mix}} \leq 12n^2$.
   (c) Consider a directed graph $G = (V, \vec{E})$ where each vertex $x \in V$ has the same indegree, $\deg_{\text{in}}(x)$, (number of edges from some y oriented towards x) and outdegree, $\deg_{\text{out}}(x)$, (number of edges from x oriented to some y). Let $t_{\text{mix}}^{(\infty)}$ be the $\mathcal{L}^\infty$ mixing time for a lazy simple random walk on G, which is defined by $P(x, x) = \frac{1}{2}$ for all $x \in V$ and $P(x, y) = \frac{1}{2\deg_{\text{out}}(x)}$ for $(x, y) \in \vec{E}$ and all other transition probabilities are 0. Show that if the walk is irreducible for all $\varepsilon > 0$ we have

$$t_{\text{mix}}^{(\infty)}(\varepsilon) \lesssim \frac{1}{\varepsilon} \min\left\{mn, \frac{m^2}{\varepsilon}\right\}.$$

   (Hint: Use the spectral profile bound on the mixing time and lower bound the spectral profile by considering the additive symmetrization of P and bounding the spectral profile of it in terms of the maximal hitting times for the symmetrization chain.)

14. For any $d \geq 2$ find the order of expected maximal hitting time of the simple random walk on d-dimensional torus of side length n, denoted $\mathbb{Z}_n^d$. Deduce the order of mixing and relaxation time for the lamplighter walk on this graph. Deduce also the order of the mixing time of the lazy simple random walk on a graph obtained by connecting two copies of $\mathbb{Z}_n^d$ using one edge.

15. Let X be a lazy biased walk on $\{0, \ldots, n\}$ with $P(i, i+1) = \frac{1}{3} = \frac{1}{2} - P(i, i-1)$ for $i \in \{1, \ldots, n-1\}$ and $P(i,i) = \frac{1}{2}$ and $P(n, n-1) = P(0,1) = \frac{1}{2}$. Show that X exhibits cutoff at $6n$, i.e., that for all $\varepsilon \in (0,1)$ we have as $n \to \infty$

$$t_{\text{mix}}(\varepsilon) = 6n(1 + o(1)).$$

16. Show that the following Markov chain on $S = \{0, 1, \ldots, 2n+1\} \cup \{-10n, -10n-2, \ldots, -2\}$, i.e. set of all numbers between 0 and $2n+1$ and even numbers between $-10n$ and 0 with the non-zero transition probabilities as follows (see figure 2) is reversible, satisfies $t_{\text{rel}} \ll t_{\text{mix}}$ but does not have cutoff:

$$P_n(x,y) = \begin{cases} \frac{1}{2} & \text{for } x \text{ even and } y = x, \text{ or } (x,y) = (-10n, -10n+2) \\ \frac{3}{4} & \text{for } x < 2n+1 \text{ odd and } y = x \\ \frac{9}{10} & \text{for } x = y = 2n+1 \\ \frac{1}{5} & \text{for } (x,y) = (0,1) \text{ or } (x,y) = (0,2) \\ \frac{1}{10} & \text{for } (x,y) = (0,-2) \\ \frac{1}{20} & \text{for } (x,y) = (2n+1, 2n) \text{ or } (x,y) = (2n+1, 2n-1) \\ \frac{1}{6} & \text{for } (x,y) = (2i, 2i-2), \text{ for } i \neq 0, -5n \leq i \leq n \\ \frac{1}{6} & \text{for } (x,y) = (2i-1, 2i+1) \text{ for } 1 \leq i \leq n \\ \frac{1}{3} & \text{for } (x,y) = (2i, \min\{2i+2, 2n+1\}), \text{ for } -5n \leq i \leq n, i \neq 0 \\ \frac{1}{12} & \text{for } (x,y) = (2i-1, \max\{2i-3, 0\}) \text{ for } 1 \leq i \leq n \end{cases}$$

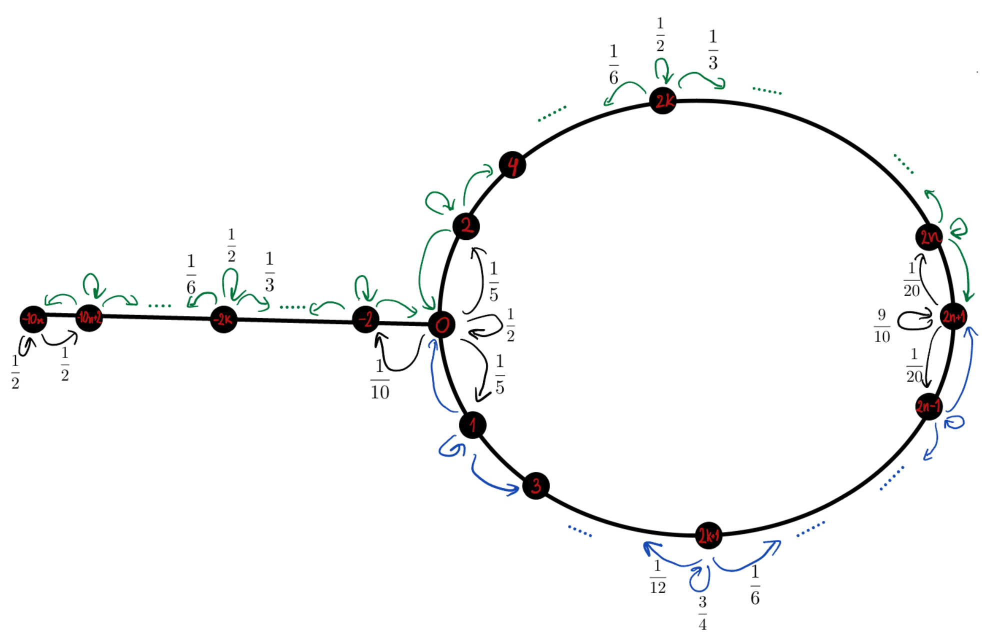

Figure 2: Markov chain from Question 16.