Example Sheet 3

Questions marked with * are optional. Please hand in Questions 2, 8 and 13.

1. Let X be a reversible Markov chain on the finite state space S with invariant distribution π. Let $A \subseteq S$ and set $T_A^+ = \min\{t \geq 1 : X_t \in A\}$ to be the first return time to A. Consider now the induced chain on A, i.e., the chain with transition matrix

$$P_A(x, y) = \mathbb{P}_x\left(X_{T_A^+} = y\right) \text{ for } x, y \in A.$$

   (a) Find the invariant distribution of the chain with matrix P_A and show that it is reversible.

   (b) Let $\varphi : A \to \mathbb{R}$ be a function and define $\widetilde{\varphi}(x) = \mathbb{E}_x[\varphi(X_{T_A})]$. Show that $P\widetilde{\varphi} = P_A\varphi$.

   (c) Let γ_A be the spectral gap of the chain with matrix P_A and γ the spectral gap of P. Prove that $\gamma_A \geq \gamma$.

2. Let $A \subseteq [0, n]^2 \cap \mathbb{Z}^2$ be obtained by removing the vertices of $[0, n]^2 \cap \mathbb{Z}^2$ with both coordinates even. Consider a simple random walk on A and let γ_A be its spectral gap. Using comparison results or otherwise, prove that there exists a positive constant c so that

$$\gamma_A \geq \frac{c}{n^2}.$$

3. Let π be a distribution on the finite set S. Consider the transition matrix $P(x, y) = \pi(y)$ for all $x, y \in S$.

   (a) Show that the spectral gap of the matrix P is equal to 1.

   (b) Consider the Markov chain on S^n which at every step chooses one coordinate at random and moves it according to the matrix P. Let γ be its spectral gap. Show that $\gamma = \frac{1}{n}$.

   (c) Deduce the Effron-Stein inequality: let $(X_1, \ldots, X_n)$ and $(X_1', \ldots, X_n')$ be i.i.d. vectors distributed according to $\pi \times \cdots \times \pi$. Let $f : S^n \to \mathbb{R}$ be a function. Show that

$$\mathrm{Var}\left(f(X)\right) \leq \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}_X\left[\left(f(X_1, \ldots, X_n) - f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)\right)^2\right].$$

4. Show that for any transition matrix P on S with invariant distribution π, and $A \subset S$ we have $Q(A, A^c) = Q(A^c, A)$, where $Q(A, B) = \sum_{x \in A, y \in B} Q(x, y)$ and $Q(x, y) = \pi(x)P(x, y)$ for $x, y \in S$. Use this to show that $\Phi_*(P) = \Phi_*(P^*) = \Phi_*(\frac{P+P^*}{2})$, where $\Phi_*(R)$ is the bottleneck ratio corresponding to transition matrix R and P^* is a time reversal of P.

5. * Show that there exists $\delta \in (0, 1)$ sufficiently small such that

$$\sum_{k=1}^{n/2} \binom{n}{\delta k} \frac{\binom{(1+\delta)k}{k}^2}{\binom{n}{k}} \to 0 \text{ as } n \to \infty,$$

   thus completing the proof of Theorem 5.25 .

6. Let P be a transition matrix on S and recall that Λ is the spectral profile, Φ_* is the isoperimetric profile, $\gamma(P)$ is the Poincaré constant, Φ is the bottleneck ratio and for $A \subset S$ $\lambda_0(A) = \inf_{f \in c_0(A)} \frac{\mathcal{E}_{P_A}(f)}{\|f\|_2^2} = \inf_{f \in c_0(A)} \frac{\langle (I - P_A)f, f\rangle_\pi}{\|f\|_2^2}$ where $P_A(i, j) = P(i, j)$ for all $i, j \in A$ and 0 otherwise, and that for $r \geq \pi_{\min}$ we defined $\Lambda_0(r) = \inf_{\pi(A) \leq r} \lambda_0(A)$. Show that:

(a) $\lambda_0(A) \le \lambda(A) \le \frac{1}{1-\pi(A)}\lambda_0(A)$ and therefore $\Lambda_0(r) \le \Lambda(r) \le \frac{1}{1-r}\Lambda_0(r)$

(b) $\Lambda\left(\frac{1}{2}\right) \le 2\gamma(P)$

(c) when P is reversible $\lambda_0(A)$ is also the smallest eigenvalue of matrix $(I-P_A)_{ij\in A}$

(d) $\frac{1}{2}\Phi_*^2(\pi(A)) \le \lambda_0(A) \le \Phi(A)$ and conclude the generalised Cheegers inequality saying $\frac{\Phi_*^2(r)}{2} \le \Lambda(r) \le \frac{\Phi_*(r)}{1-r}$.

7. Show that for a lazy Markov chain X on S with invariant distribution π and distribution μ on S we have
$$\|\mathbb{P}_\mu(X_t \in \cdot) - \pi\|_2^2 \le M, \text{ if } t > 2\left\lceil \int_{\frac{4}{\|\mu-\pi\|_2^2}}^{\frac{4}{M}} \frac{du}{u\Lambda(u)} \right\rceil.$$

8. Let G be a graph obtained by adding some edges to the two-dimensional box of side length n so that the maximal degree of every vertex is at most some constant $\Delta > 4$. Let t_{mix} be the mixing time of a lazy simple random walk on G.

   (a) Prove that $t_{\text{mix}} \lesssim n^2$.

   (b) Show that there exists an example of such a graph G (i.e. box with added edges in such a way that the degrees are bounded) such that $t_{\text{mix}} \asymp \log n$.

   (c) By upper bounding the spectral gap using variational characterisation, or otherwise, show also that if $d_n > 0$ is such that any added edge connects vertices which have graph distance in the box at most d_n (i.e. sum of distances of their first and second coordinates is $\le d_n$) then $t_{\text{mix}} \gtrsim \frac{n^2}{d_n^2}$.

9. Let t_{mix} be the mixing time of the lazy simple random walk on a graph G. In this question, we consider graphs G obtained by taking two bounded degree graphs G_1 and G_2 (whose sizes depend on the parameter n) and adding $f_n > 0$ edges between them such that the degrees of G stay bounded by a constant $\Delta > 0$ (not depending on n). Show that the following holds:

   (a) If G_1 and G_2 are expanders on n vertices, then $t_{\text{mix}} \lesssim \max\{\frac{n^2}{f_n^2}, \log n\}$ (regardless of how the f_n edges are added, as long as degrees are at most Δ). Conclude that there exists a sequence of bounded degree graphs which are not expanders such that the nth graph has $\asymp n$ vertices and the mixing time of a lazy walk on it is of order $\log n$.

   (b) If G_1 and G_2 are two-dimensional boxes of side length n and there exists a constant $c > 0$ (which does not depend on n) such that $f_n \ge cn^2$, then $t_{\text{mix}} \lesssim n^2$ (regardless of how the f_n edges are added, as long as degrees are at most Δ).

   (c) Let $c > 0$ be a constant and let G_1 and G_2 both be copies of two-dimensional boxes $[0,n]^2$ of side length n and let G be a random graph obtained by adding edges between them at random as follows. For each vertex $(x_1, x_2) \in [0,n]^2$ with probability $\frac{c}{n}$ add an edge between the vertex corresponding to (x_1, x_2) in G_1 and the vertex corresponding to (x_1, x_2) in G_2 independently of other edges. Show that for the random graph G obtained in this way, we have that $t_{\text{mix}} \asymp n^2$ with probability which converges to 1 as $n \to \infty$.

10. Let $M > 0$ be a constant. Show that if one changes (adds or deletes) at most M edges from all graphs in an (Δ, α, c)-small set expander family, in such a way that the graphs obtained stay connected, then the family of graphs obtained is still a small set expander. When adding edges, it is allowed to add vertices (as long as they are connected to the rest of the graph by some new edge), and if all edges from a vertex are deleted, then the vertex is also deleted (so that the graph remains connected).

11. Show that a $(\Delta, \alpha, c)-$small set expander $G = (V, E)$ can be decomposed into expanders of comparable sizes, i.e. show that there exists constants $k, \delta, \varepsilon > 0$ (depending only on Δ, α, c) and sets $V_1, V_2, \ldots, V_k \subset V$ such that $V = \bigcup_{i=1}^k V_i$ and V_i are disjoint subsets satisfying that $\inf_{i \le k} |V_i|/|V| \ge \delta$ and that $G_i = (V_i, E_i)$ is a (Δ, ε)-expander, where $E_i \subset E$ are all the edges with both endpoints in V_i.

12. Let Δ, k be fixed integers not depending on n and let $c \in (0,1)$ be a positive constant which also does not depend on n. Let $G = (V, E)$ be a Δ regular graph with $|V| = kn$ and let $V_1, \ldots, V_k$ be a partition of V into disjoint sets of size n. Let P be a transition matrix of a simple random walk on G. For all $i \le k$ assume that P_i is a transition matrix on V_i satisfying $P_i(x, y) = P(x, y)$ for $x \ne y$ and $P_i(x, x) = \sum_{y \notin V_i} P(x, y)$. Assume that for all $i \le k$, the spectral gap of P_i, γ_i satisfies $\gamma_i \ge c$. Let γ be the spectral gap of P, and let $\widetilde{\gamma}$ be the spectral gap of a transition matrix $\widetilde{P}$ on $\{1, \ldots, k\}$ with transitions $\widetilde{P}(i, j) = \sum_{x \in V_i, y \in V_j} \frac{1}{n} P(x, y)$ for all $i, j \le k$. Let π_i for $i \le k$, π and $\widetilde{\pi}$ be invariant distributions of P_i, P and $\widetilde{P}$, respectively and let π_i^j for $i \ne j$ be distribution on V_i defined by $\pi_i^j(x) := \frac{\pi_i(x) \sum_{y \in V_j} P(x,y)}{\widetilde{P}(i,j)}$. For any non-constant function f on V show that

$$\sum_{i=1}^k (\mathbb{E}_{\pi_i}[f] - \mathbb{E}_\pi[f])^2 \le \frac{3}{2\widetilde{\gamma}} \sum_{i \ne j} \widetilde{P}(i,j)[2(\mathbb{E}_{\pi_i}[f] - \mathbb{E}_{\pi_i^j}[f])^2 + (\mathbb{E}_{\pi_i^j}[f] - \mathbb{E}_{\pi_j^i}[f])^2],$$

$$\sum_{i \ne j} \widetilde{P}(i,j)(\mathbb{E}_{\pi_i^j}[f] - \mathbb{E}_{\pi_j^i}[f])^2 \le \sum_{i \ne j} \sum_{x \in V_i, y \in V_j} \frac{1}{n} P(x,y)(f(x) - f(y))^2 \quad \text{and}$$

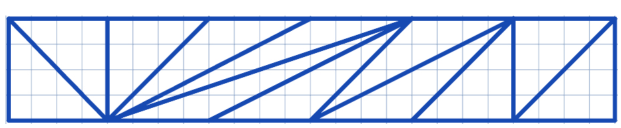
$$\sum_{i \ne j} \widetilde{P}(i,j)(\mathbb{E}_{\pi_i}[f] - \mathbb{E}_{\pi_i^j}[f])^2 \le \sum_{i \le k} \mathrm{Var}_{\pi_i}(f),$$

and hence or otherwise show that $\gamma \gtrsim c\widetilde{\gamma}$. By considering an appropriate function, show also that $\gamma \lesssim \widetilde{\gamma}$. Show that the mixing time of $\frac{I+P}{2}$ satisfies $t_{\text{mix}} \lesssim \log n + \frac{1}{\widetilde{\gamma}}$ and that if $\widetilde{\gamma} \lesssim \frac{1}{\log n}$ then $\frac{I+P}{2}$ does not exhibit cutoff.

13. Consider the exclusion process on the complete graph K_{2n} on $2n$ vertices: there are n indistinguishable black particles and n indistinguishable white particles. Each vertex is occupied by exactly one particle. At each time step, an edge is chosen at random, and the particles at its endpoints are swapped. Using path coupling or otherwise, prove an upper bound on the mixing time of order $n \log n$.

14. * Let G be a graph of maximum degree Δ. Let $q \ge \Delta + 2$. Show that the Glauber dynamics on the space of proper q-colourings is an irreducible Markov chain.

15. Let $\lambda < 1$ be a positive constant. A triangulation of $n \times 1$ rectangle, with vertices $(0,0)$, $(n,0)$, $(0,1)$, $(n,1)$ is a partition of the rectangle into $2n$ triangles such that each triangle has area $\frac{1}{2}$ and has one side length either $(x,0), (x+1,0)$ or $(x,1), (x+1,1)$ for $0 \le x \le n-1$. Here is an example of a 6×1 triangulation.



Consider a Markov chain on triangulations of $n \times 1$ rectangle with the following transitions: pick a uniformly at random value $x \in \{1, \ldots 2n-1\}$; look at the line in triangulation which is passing through the point $(\frac{x}{2}, \frac{1}{2})$ and if that line of the triangulation connects $(a, 0)$ and $(b, 1)$ then do the following

- if swapping this line to a line connecting $(a-1, 0), (b+1, 1)$ gives a valid triangulation, then make this swap with probability
$$\frac{\lambda^{|2+b-a|}}{\lambda^{|b-a|} + \lambda^{|2+b-a|}},$$
and otherwise do nothing
- if swapping the line to a line connecting $(a+1, 0), (b-1, 1)$ gives a valid triangulation, then make this swap with probability
$$\frac{\lambda^{|b-a-2|}}{\lambda^{|b-a|} + \lambda^{|b-a-2|}},$$
and otherwise do nothing
- if neither of the two swaps are possible, then do nothing.

Show that the mixing time of the above chain is at most of order n^2.

[Hint: Use the path coupling technique where the distance between triangulations which differ in one edge only is set to be 1 if the edge where they differ has unit horizontal length in both, while it is α^ℓ if the edge where they differ has horizontal length ℓ in one triangulation, while it has $\ell + 2$ in the other. The horizontal length of the edge is just the absolute value of the difference between the first coordinates of its endpoints.]

16. Let $\preceq$ be a partial order on $\{1, 2, \ldots, n\}$[1]. A linear extension of $\preceq$ is a total order[2], $\precsim$ on $\{1, 2, \ldots n\}$ that respects $\preceq$, i.e. for all $i, j \in \{1, \ldots, n\}$, $i \preceq j$ implies $i \precsim j$. Let S be the set of all linear extensions of $\preceq$. Consider a Markov chain on S with transitions from some $\precsim$ from S, for which $a_1 \precsim a_2 \precsim a_3 \ldots \precsim a_n$, defined as follows. First, let p be chosen at random from the set $\{1, \ldots n-1\}$ such that $p = i$ with probability $\frac{i(n-i)}{\frac{1}{6}(n^3-n)}$ and let $c \in \{0, 1\}$ be picked uniformly at random. If $c = 0$ the chain stays in place, while if $c = 1$ if swapping the order of a_p and a_{p+1} still gives a linear extension of $\preceq$ then the two elements are swapped i.e. the chain is moved to the total order $\precsim_1$ for which $a_1 \precsim_1 a_2 \precsim_1 \ldots a_{p-1} \precsim_1 a_{p+1} \precsim_1 a_p \precsim_1 a_{p+2} \ldots \precsim_1 a_n$, and if this transition does not give a linear extension of $\preceq$ then the chain stays in place. Show that the invariant distribution of this chain is the uniform distribution on S and that the mixing time satisfies $t_{\text{mix}} \lesssim n^3 \log n$.

    [Hint: Consider a length function defined on a graph where vertices are permutations corresponding to orders in S and two permutations are connected if they only differ by a transposition.]

17. * Let G be a finite graph and consider the Glauber dynamics for the Ising model on G. Show that for all values of the inverse temperature β we have
$$t_{\text{rel}} \geq \frac{n}{2}.$$
(Hint: Use the test function which is the spin at a single vertex.)

[1] partial order is a relation which satisfies: $x \preceq x$; if $x \preceq y$ and $y \preceq z$ then $x \preceq z$; if $x \preceq y$ and $y \preceq x$ then $x = y$.

[2] total order is a partial order for which for any two elements x, y either $x \precsim y$ or $y \precsim x$; you can think of total orders as permutations of elements where if x is before y in the permutation then $x \precsim y$.